

# Digital Image Processing

## Lecture # 07

### Frequency Domain Image Analysis

# Image Enhancement in Frequency Domain

# Joseph Fourier (1768 – 1830)



- Most famous for his work “*La Théorie Analytique de la Chaleur*” published in 1822
- Translated into English in 1878: “*The Analytic Theory of Heat*”

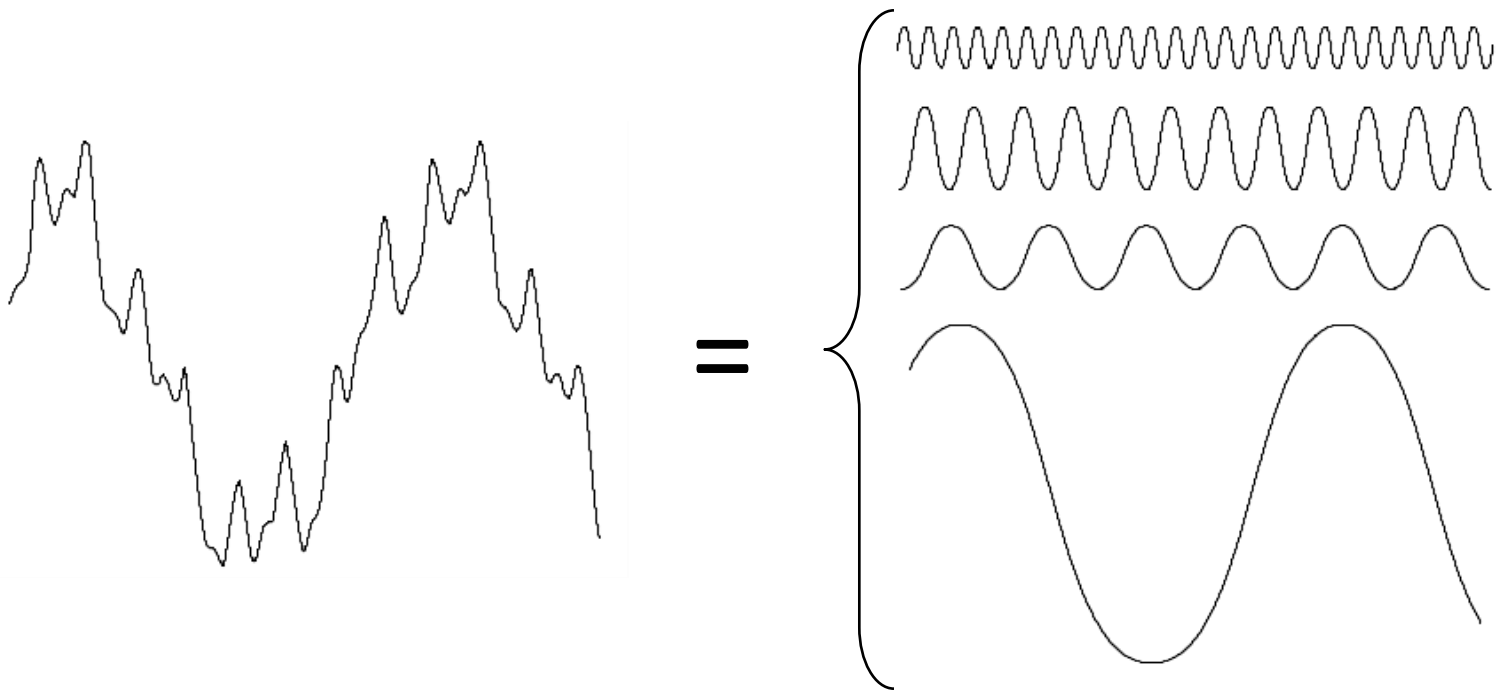
Nobody paid much attention when the work was first published  
One of the most important mathematical theories in modern engineering

# Background

- Any function that **periodically** repeats itself can be expressed as the **sum** of sines and/or cosines of different frequencies, each multiplied by a different coefficient (**Fourier series**).
- Even functions that are **not periodic** (but whose area under the curve is finite) can be expressed as the **integral** of sines and/or cosines multiplied by a weighting function (**Fourier transform**).

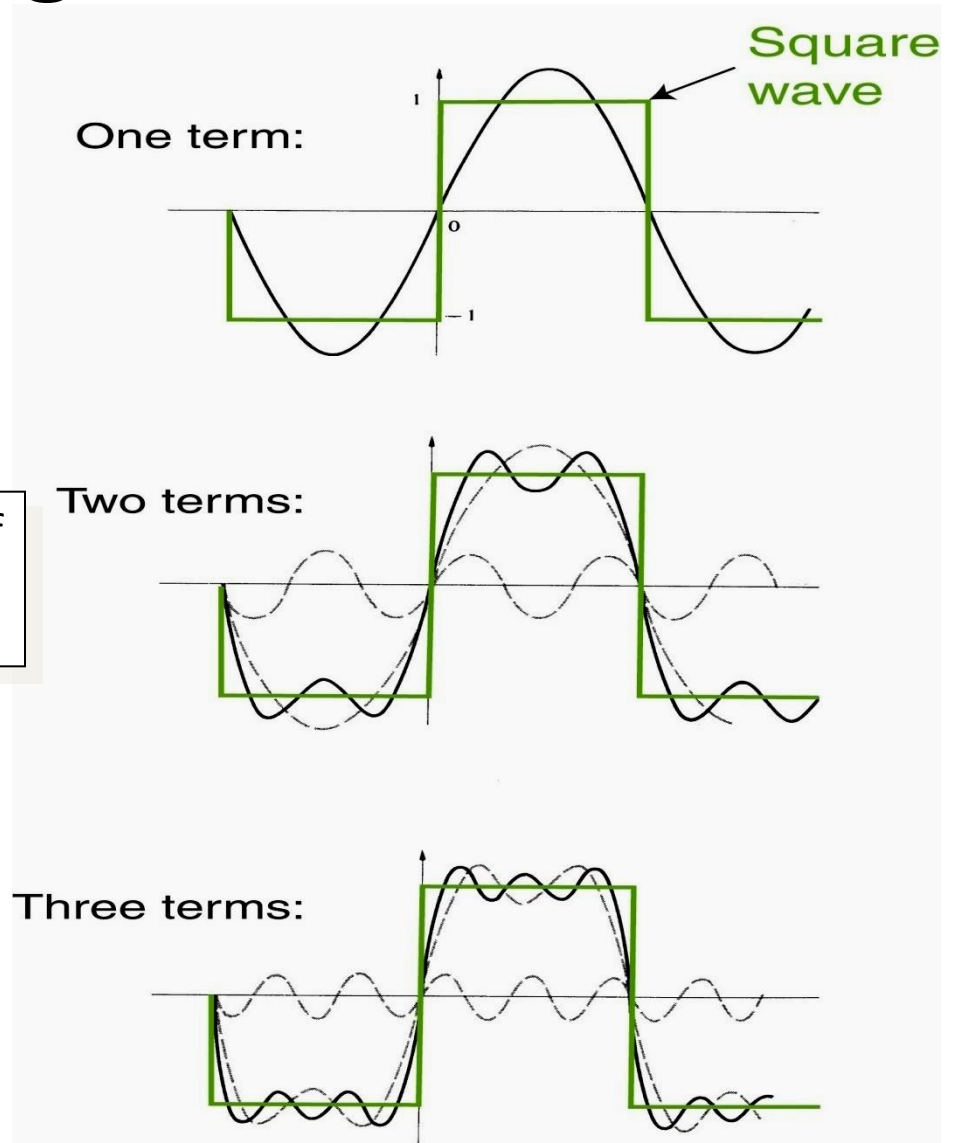
# The big idea ...

Any function that periodically repeats itself can be expressed as a sum of sines and cosines of different frequencies each multiplied by a different coefficient – a *Fourier series*

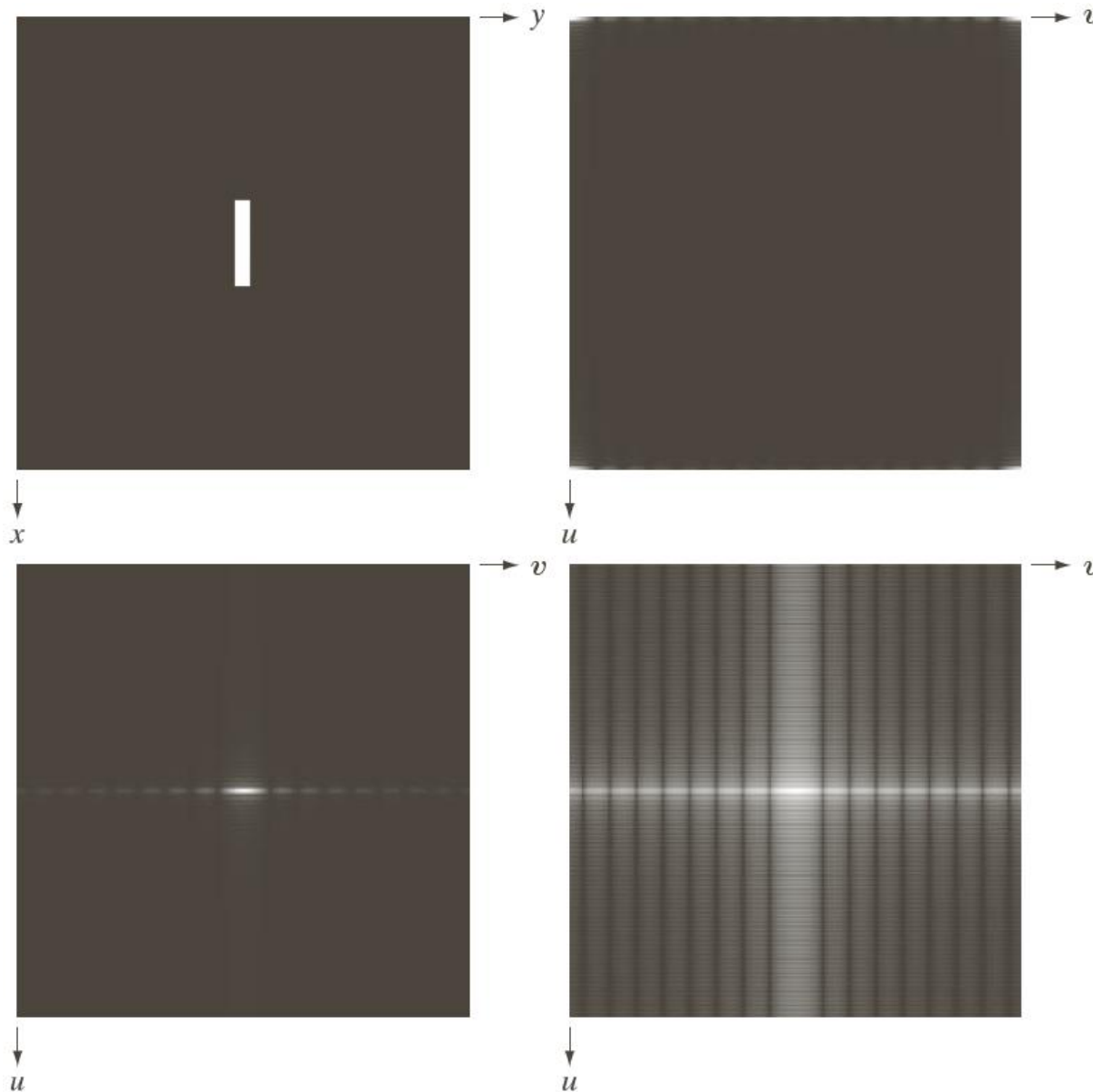


# The big idea...

Approximating a square wave as the sum of sine waves



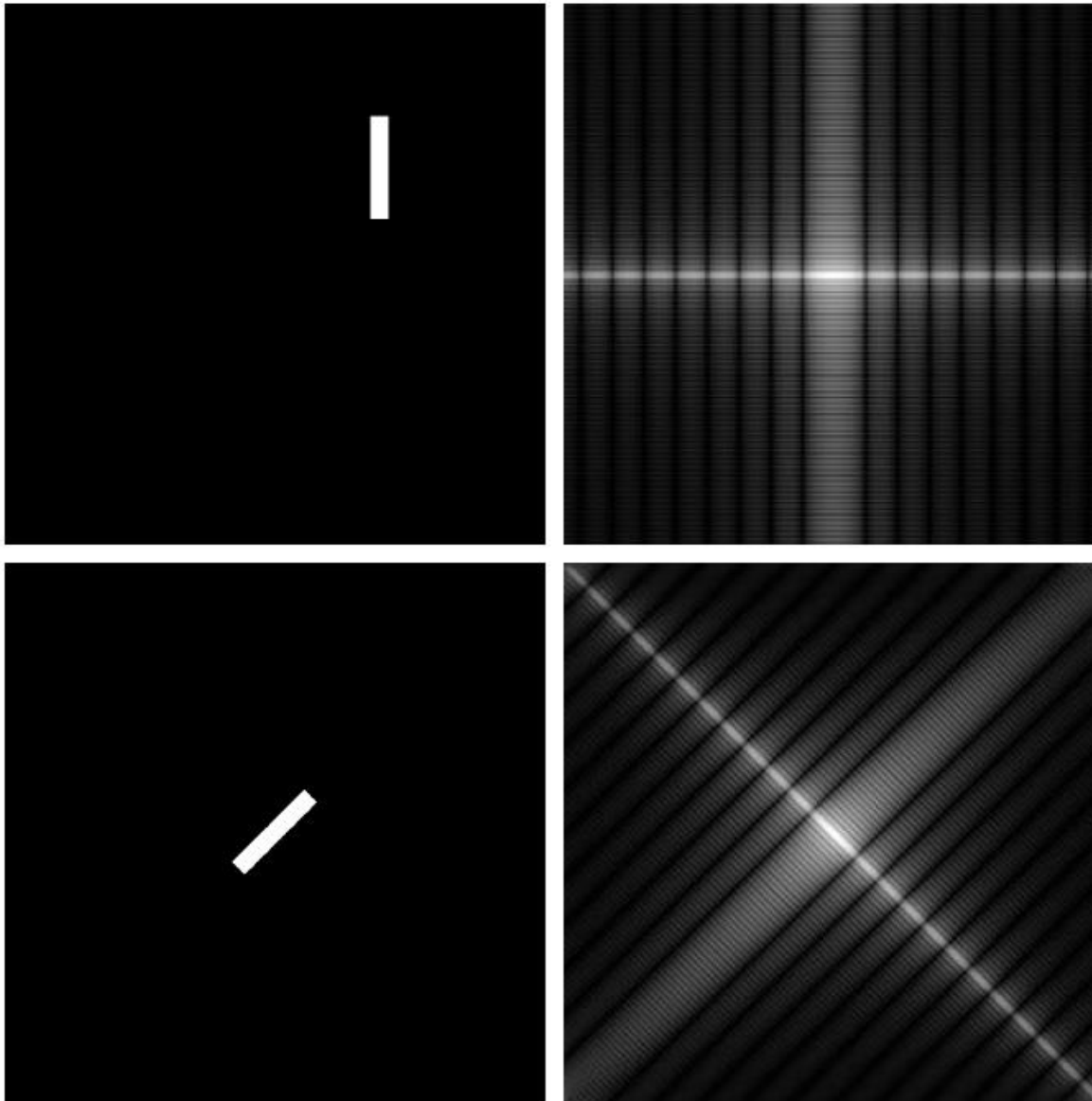
# Frequencies in Images



a	b
c	d

**FIGURE 4.24**

(a) Image.  
(b) Spectrum showing bright spots in the four corners.  
(c) Centered spectrum.  
(d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.

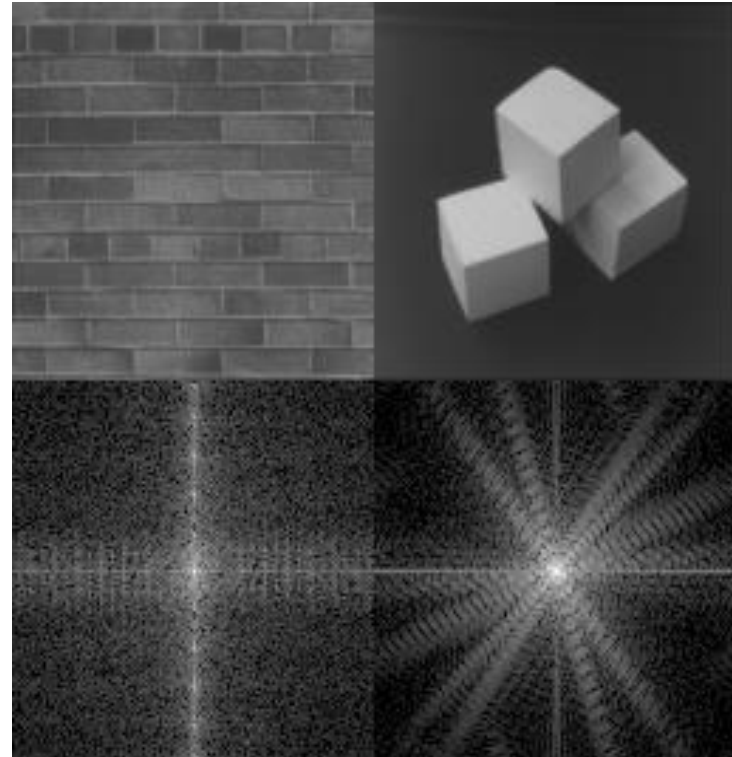
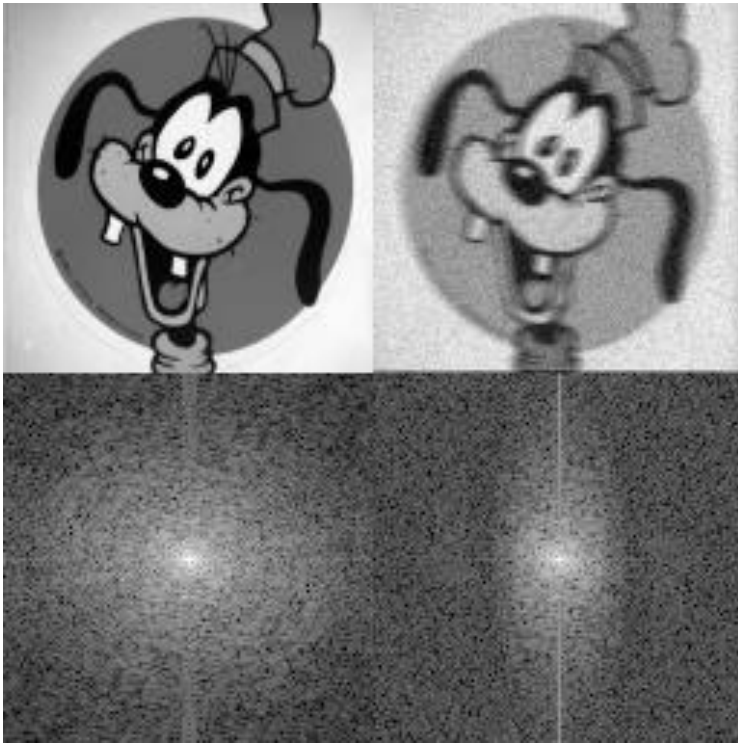


a	b
c	d

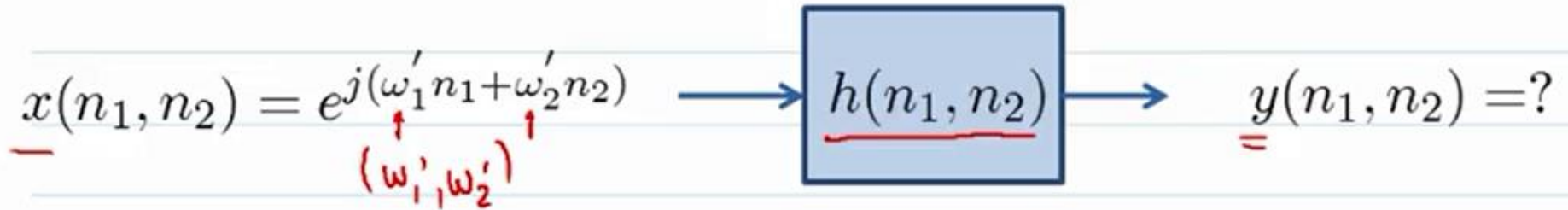
**FIGURE 4.25**

(a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).

# Frequencies in Images



# Basic 2D FT



$$y(n_1, n_2) = X(n_1, n_2) ** h(n_1, n_2)$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} e^{j\omega'_1(n_1-k_1)} e^{j\omega'_2(n_2-k_2)} h(k_1, k_2)$$

$$= \underbrace{e^{j\omega'_1 n_1} e^{j\omega'_2 n_2}}_{\text{}} \underbrace{\sum_{k_1} \sum_{k_2} h(k_1, k_2) e^{-j\omega'_1 k_1} e^{-j\omega'_2 k_2}}_{\text{}}$$

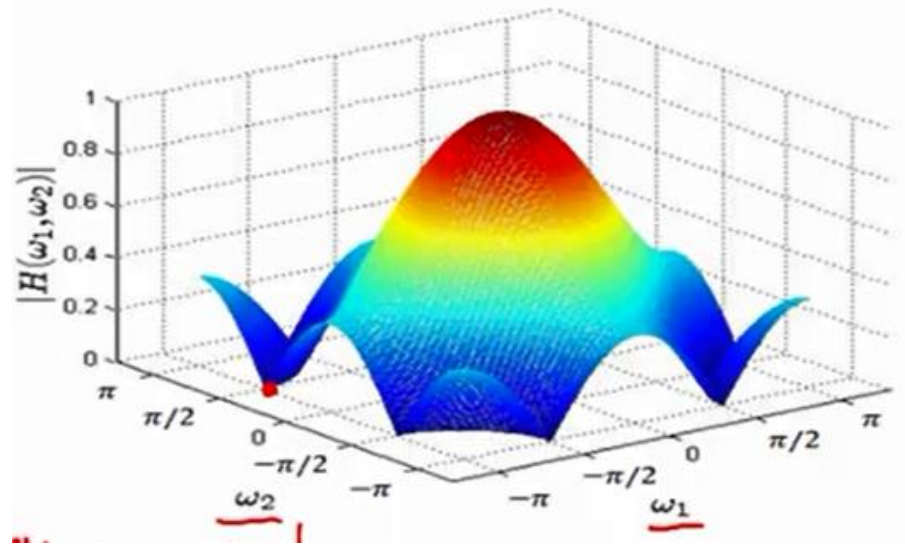
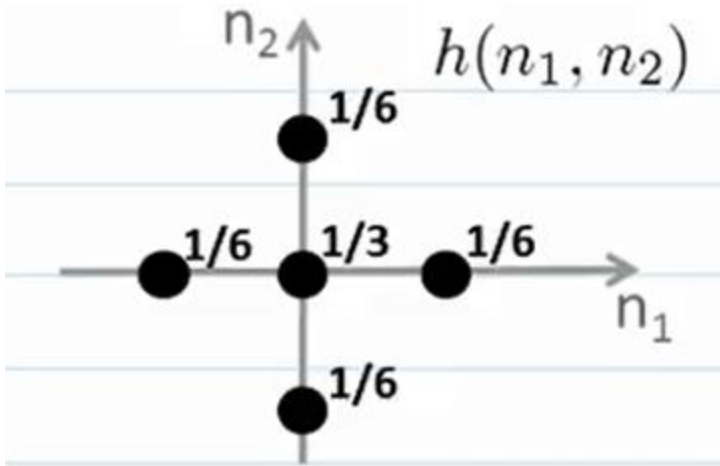
$$H(\omega'_1, \omega'_2) \triangleq \text{frequency response}$$

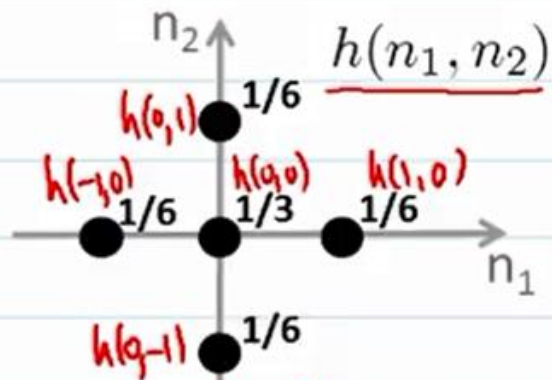
# 2D FT

$$X(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

$$x(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) e^{j\omega_1 n_1} e^{j\omega_2 n_2} d\omega_1 d\omega_2$$

# Example





$$H(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} h(n_1, n_2) e^{j\omega_1 n_1} e^{-j\omega_2 n_2}$$

$$= h(0,0) + h(-1,0)e^{j\omega_1} + h(1,0)e^{-j\omega_1} + h(0,1)e^{j\omega_2} + h(0,-1)e^{-j\omega_2}$$

$$= \frac{1}{3} + \frac{1}{6}e^{j\omega_1} + \frac{1}{6}e^{-j\omega_1} + \frac{1}{6}e^{j\omega_2} + \frac{1}{6}e^{-j\omega_2}$$

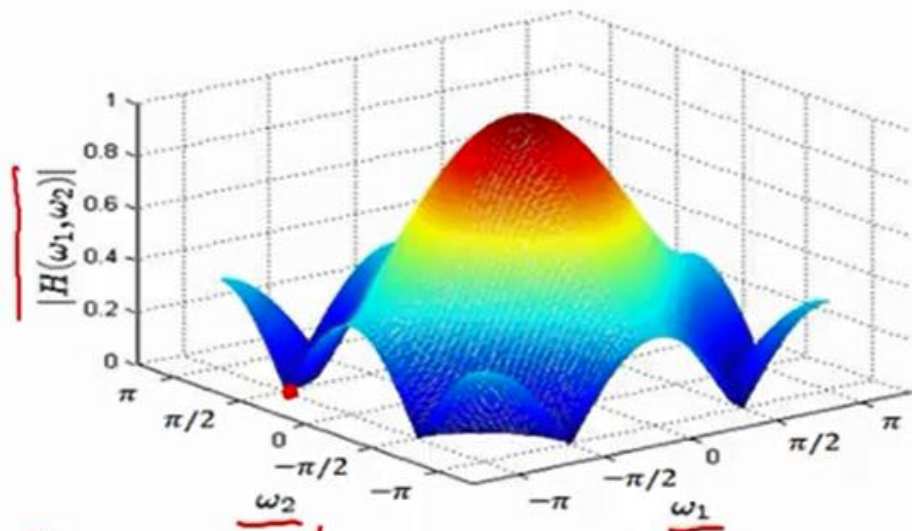
$$= \frac{1}{3} + \frac{1}{6} \cdot 2 \cos \omega_1 + \frac{1}{6} \cdot 2 \cos \omega_2$$

$$= \frac{1}{3} (1 + \cos \omega_1 + \cos \omega_2)$$

$$-1 = 1 \cdot e^{\pm j\pi}$$

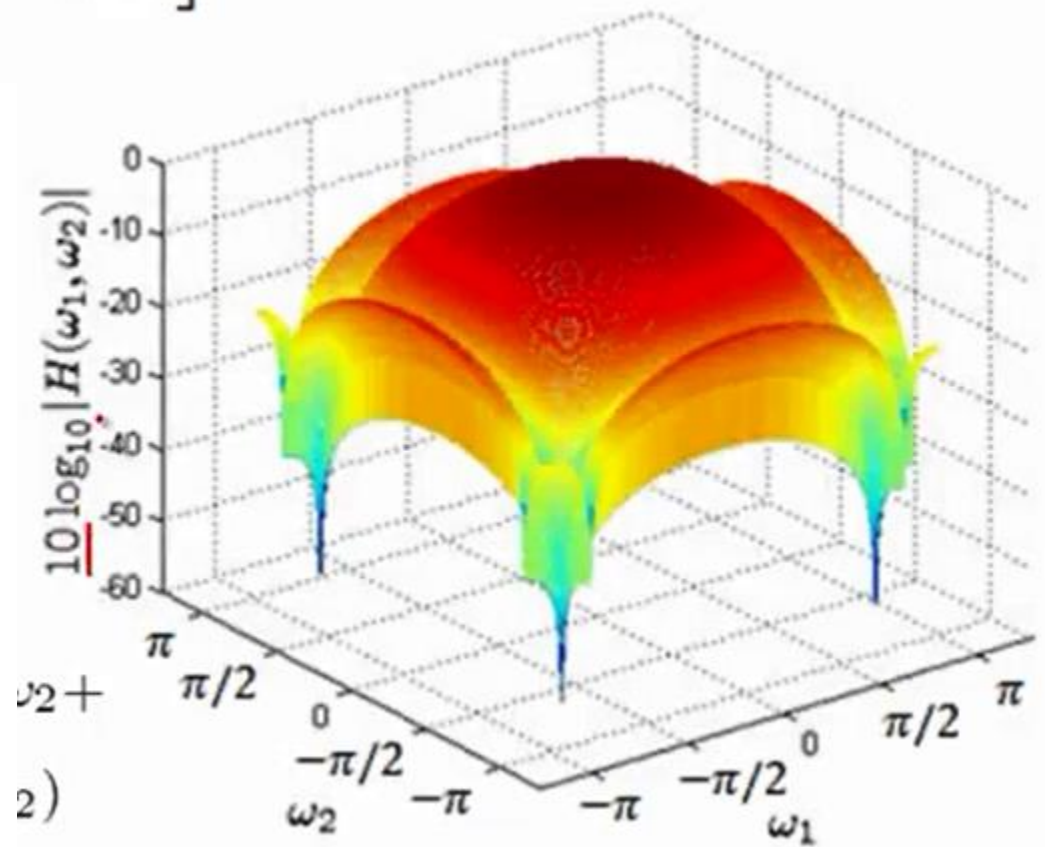
$$H(0,0) = 1$$

$$H(-\pi, \pi/2) = 0$$



# Example

$$h(n_1, n_2) = \begin{bmatrix} 0.075 & 0.124 & 0.075 \\ 0.124 & 0.204 & 0.124 \\ 0.075 & 0.124 & 0.075 \end{bmatrix}$$

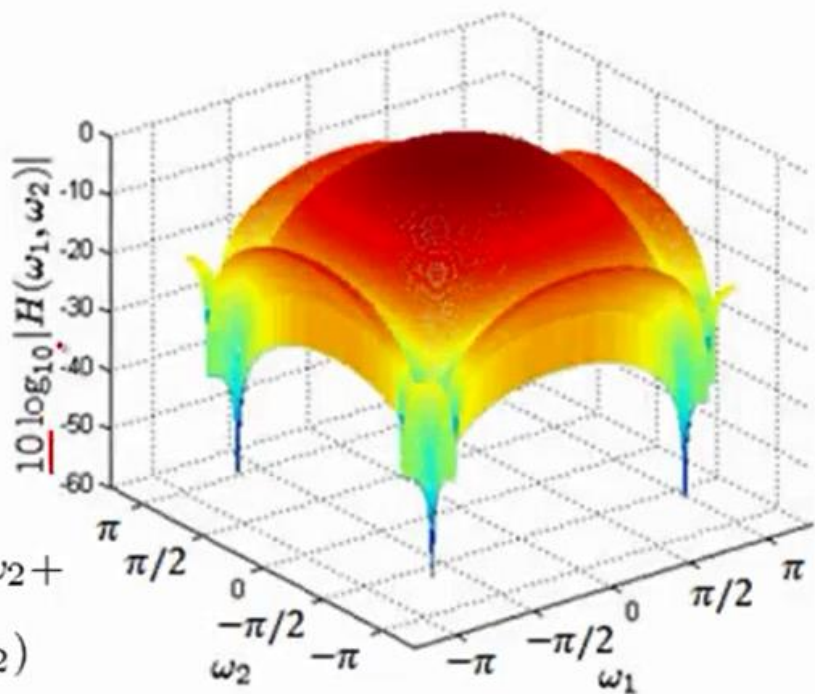


2)

$$\underline{h(n_1, n_2)} = \begin{bmatrix} \textcircled{0.075} & 0.124 & \textcircled{0.075} \\ \textcircled{0.124} & \textcircled{0.204} & \textcircled{0.124} \\ \textcircled{0.075} & 0.124 & \textcircled{0.075} \end{bmatrix}$$

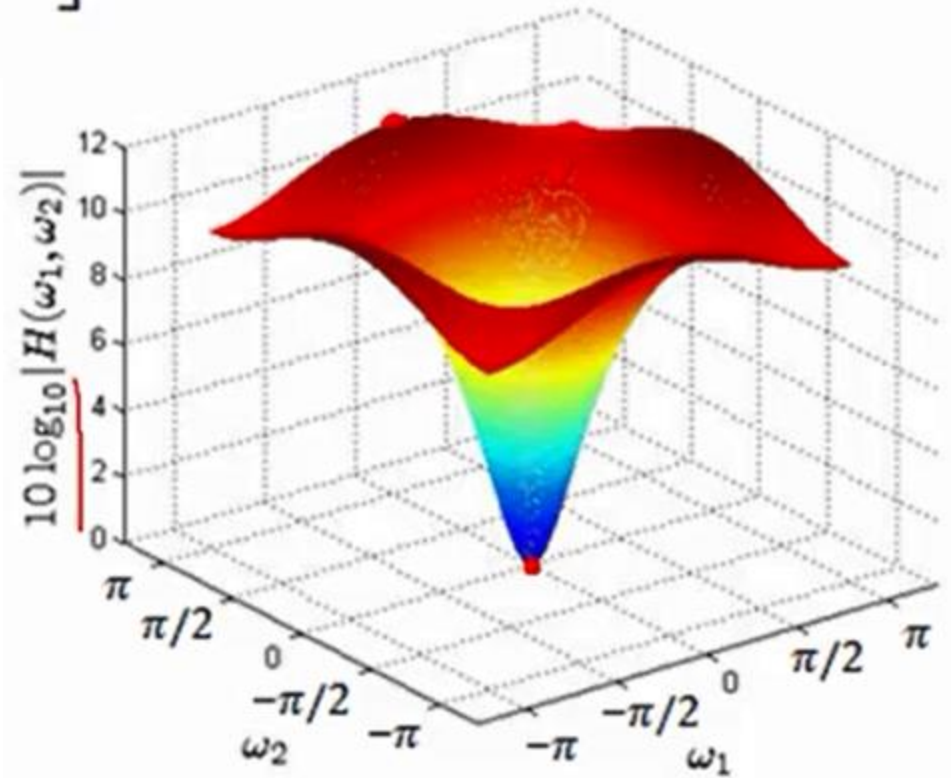
$0.124 \cdot 2 \cdot \cos \omega_1$   
 $h(-1, 1)$   $h(1, 1)$   
 $h(0, 0)$   
 $0.075 \cdot 2 \cdot \cos(\omega_1 + \omega_2)$

$$\underline{H(\omega_1, \omega_2)} = 0.204 + 0.124 \cdot 2 \cdot \cos \omega_1 + 0.124 \cdot 2 \cdot \cos \omega_2 + 0.075 \cdot 2 \cdot \cos(\omega_1 + \omega_2) + 0.075 \cdot 2 \cdot \cos(\omega_1 - \omega_2)$$



# Example

$$h(n_1, n_2) = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 9 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

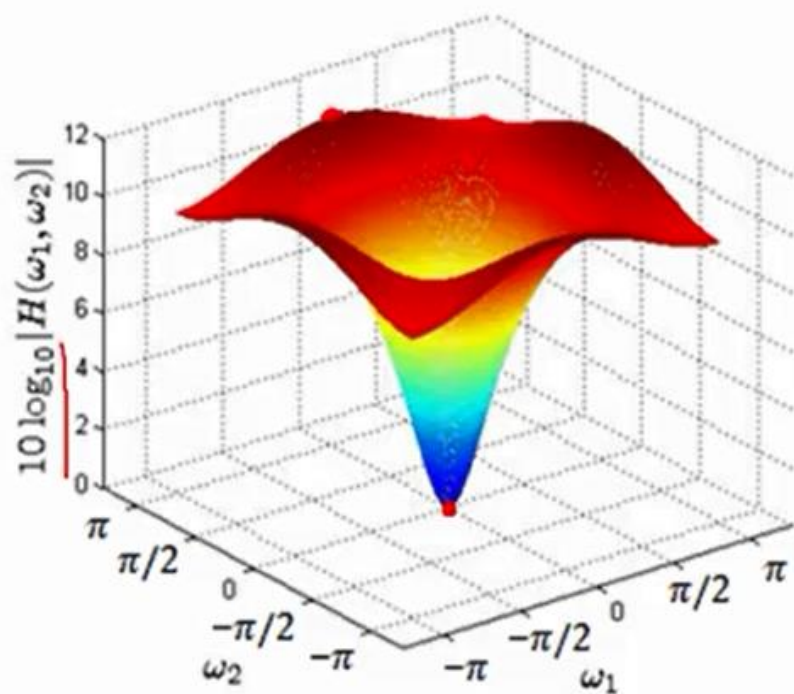


$$\underline{h(n_1, n_2)} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & \textcircled{9} & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

*h(0,0)*

$$\underline{H(\omega_1, \omega_2)} = 9 - 2 \cdot \cos\omega_1 - 2 \cdot \cos\omega_2 -$$

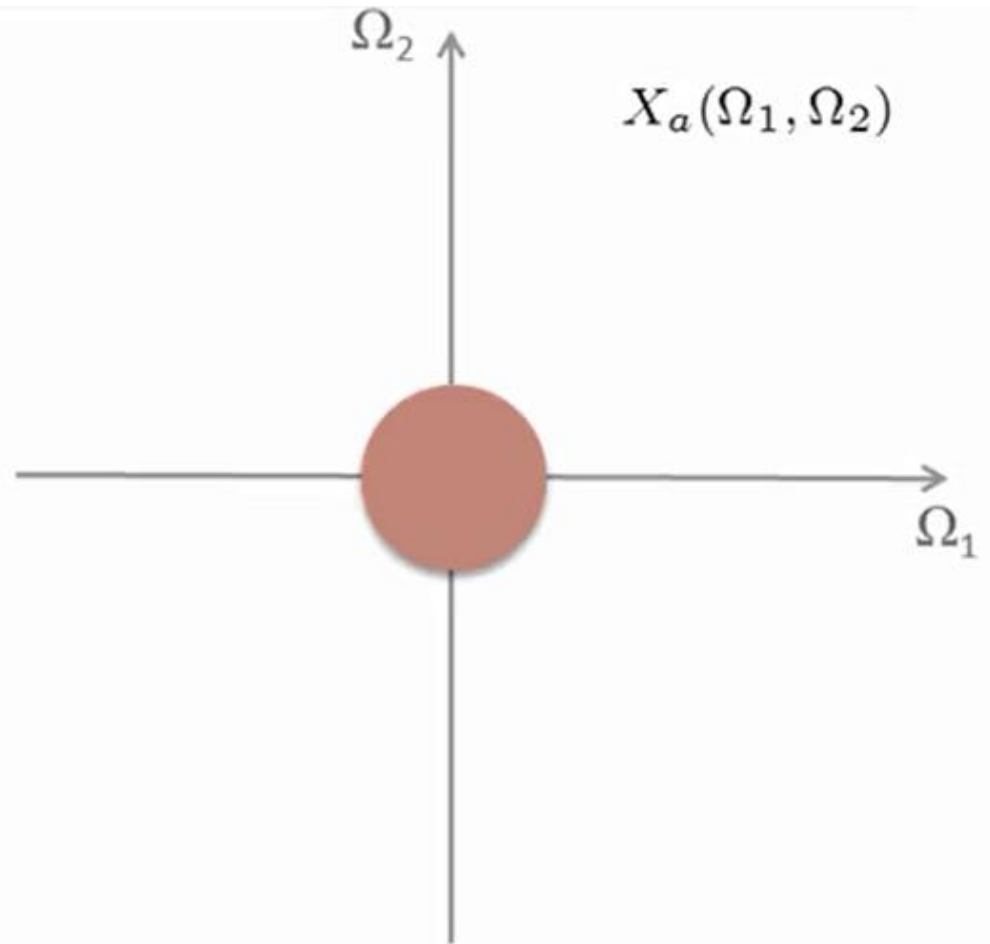
$$2 \cdot \cos(\omega_1 + \omega_2) - 2 \cdot \cos(\omega_1 - \omega_2)$$



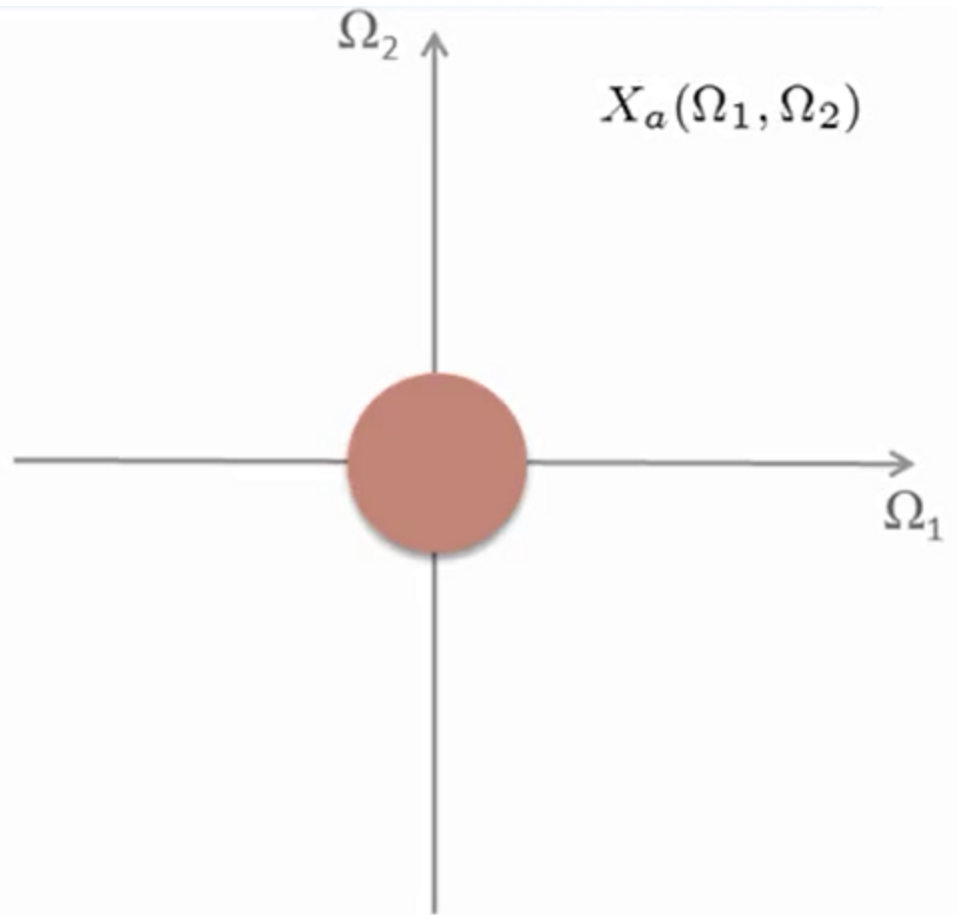
$$H(0,0) = 1 \rightarrow \log H(0,0) = 0$$

$$H(0,\pi) = 13, \quad H(\pi,\pi) = 9$$

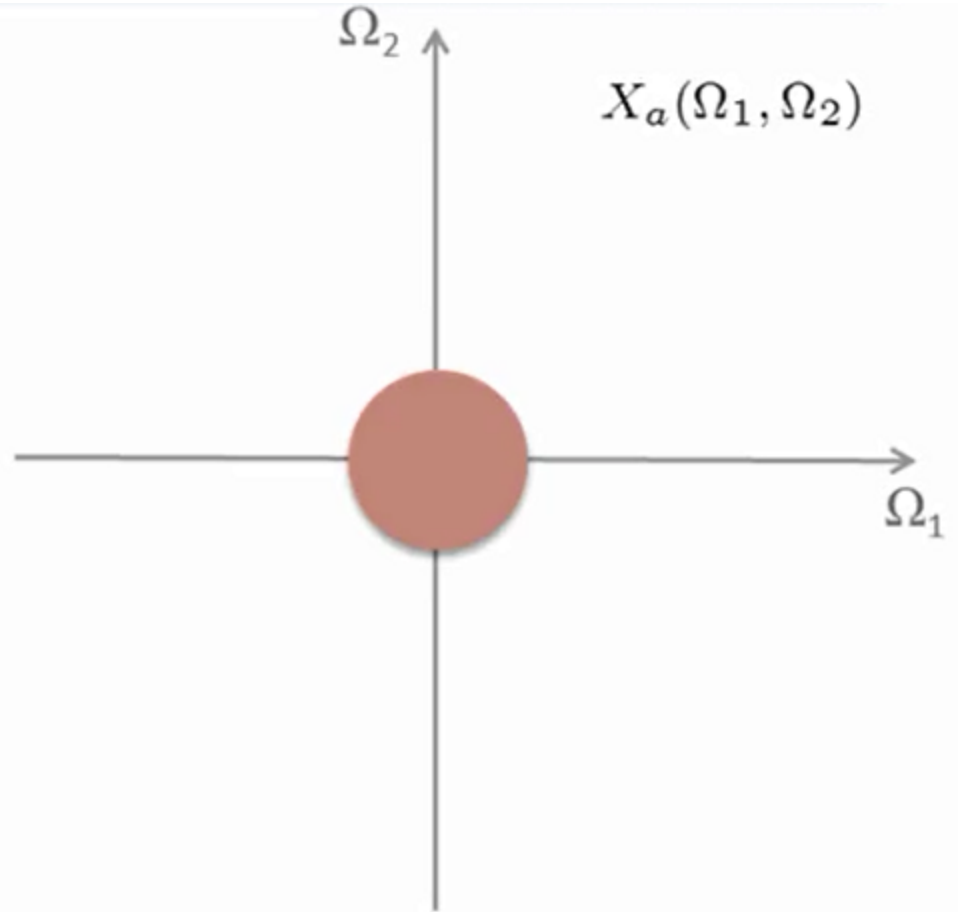
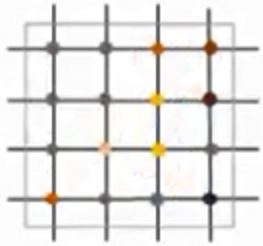
# 2D Sampling



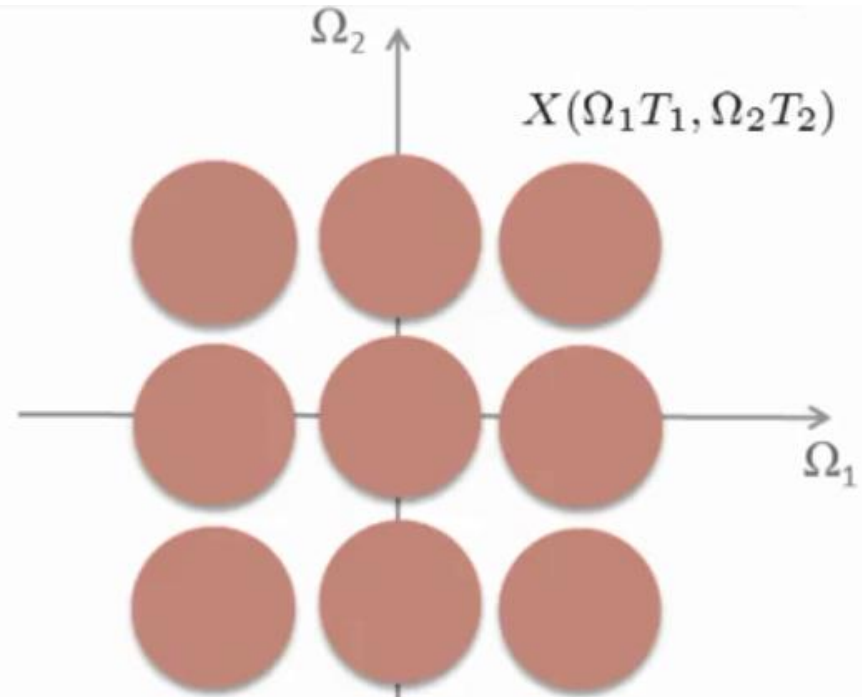
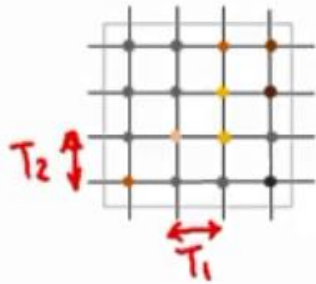
# 2D Sampling



# 2D Sampling

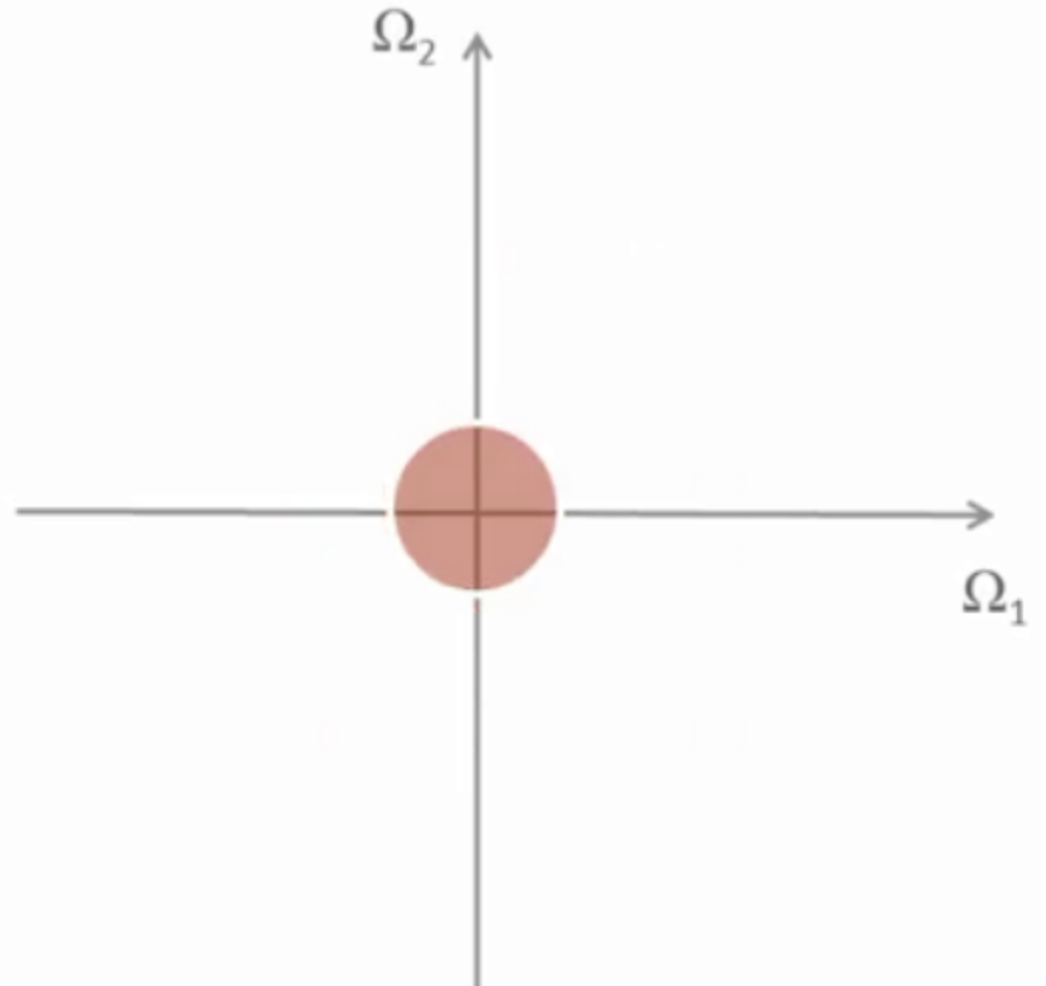


# 2D Sampling

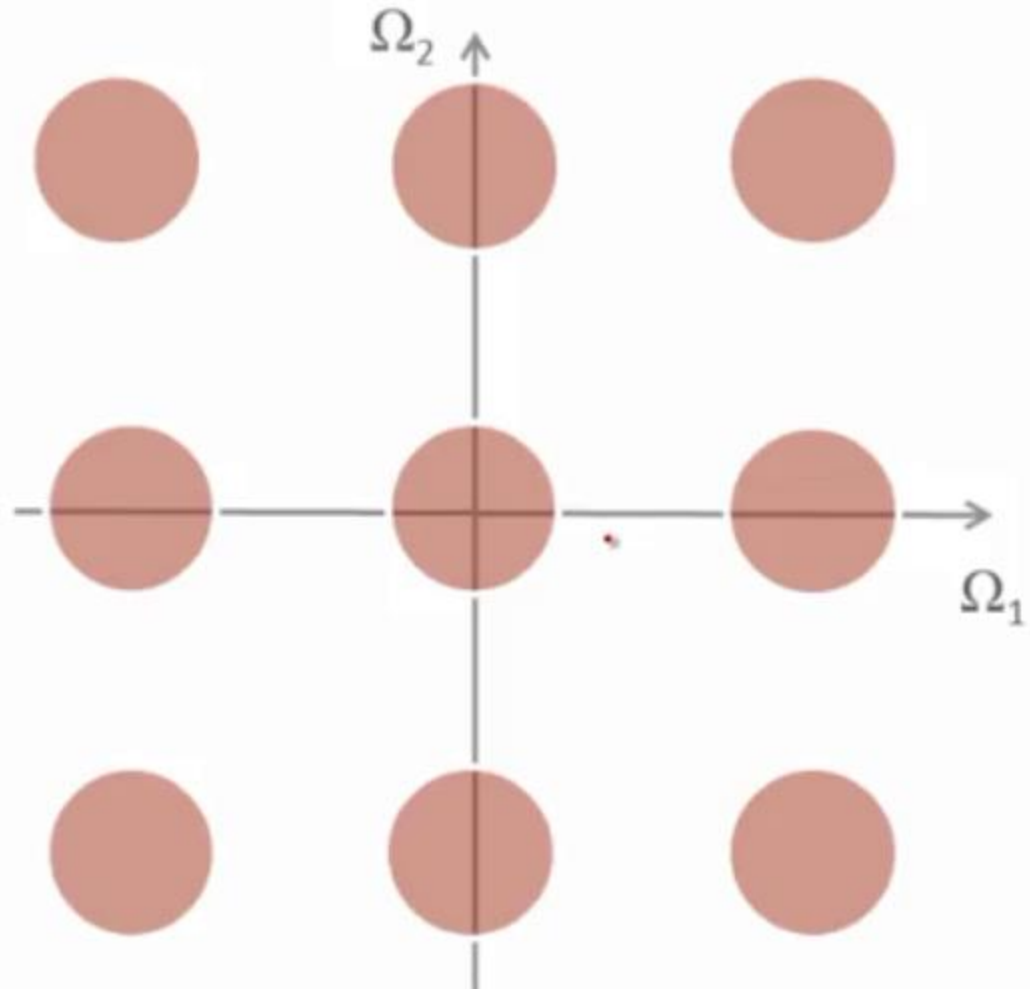
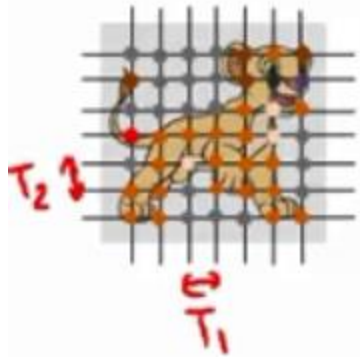


$$X(\Omega_1 T_1, \Omega_2 T_2) = \frac{1}{T_1 T_2} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} X_a\left(\Omega_1 - \frac{2\pi}{T_1} k_1, \Omega_2 - \frac{2\pi}{T_2} k_2\right)$$

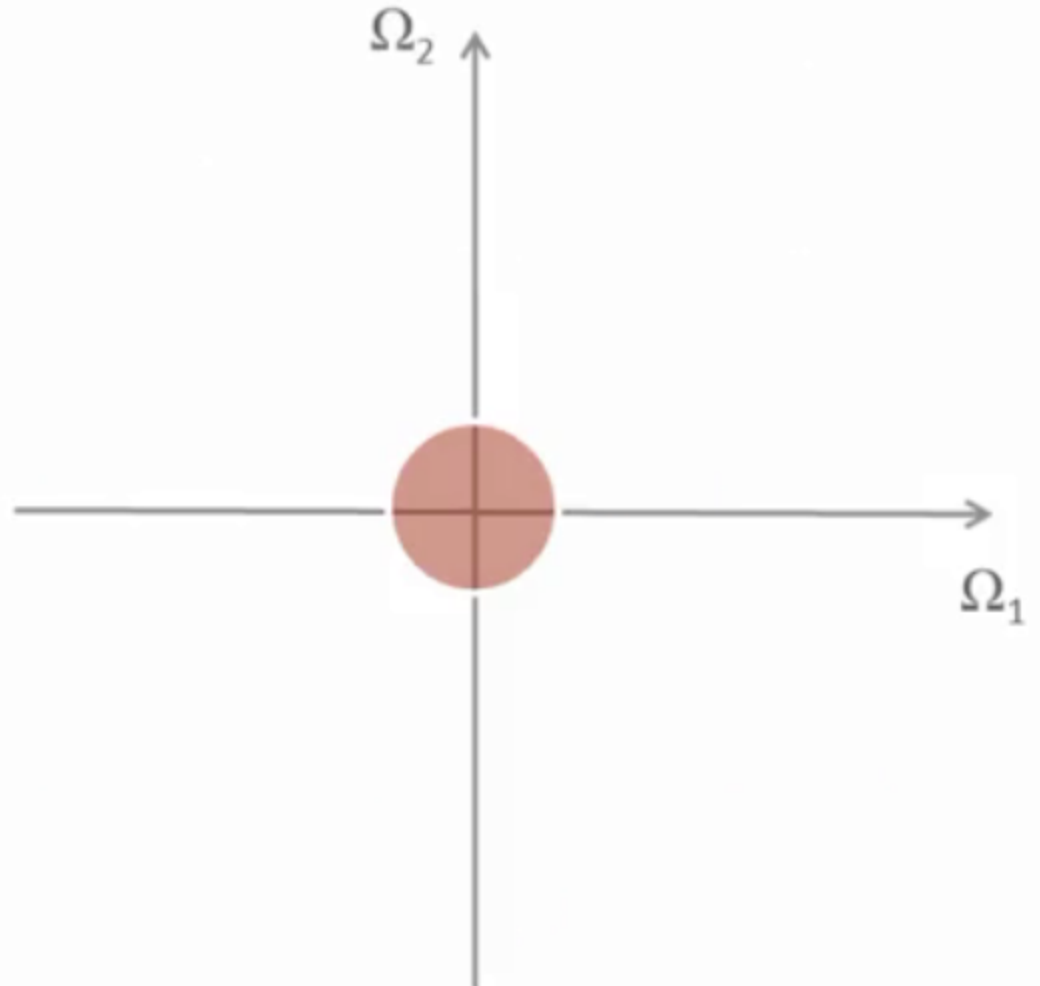
# Over Sampling



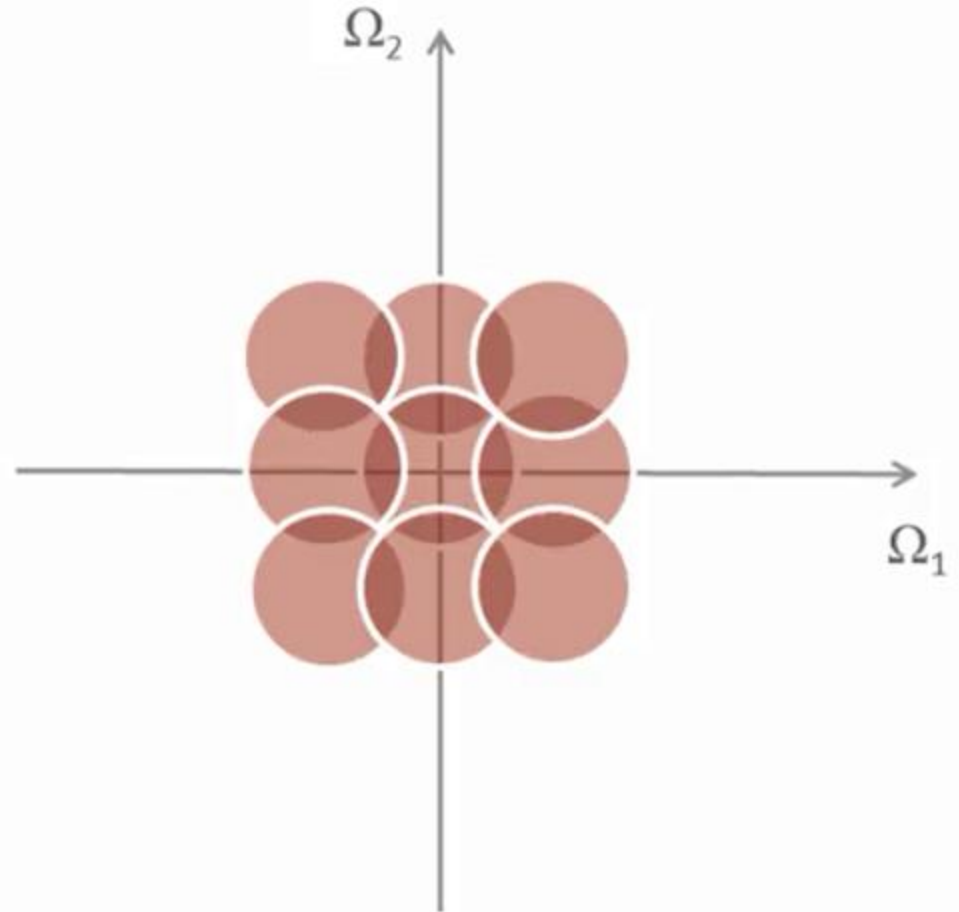
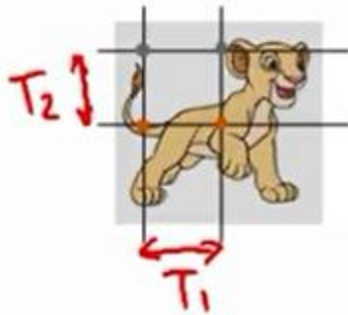
# Over Sampling



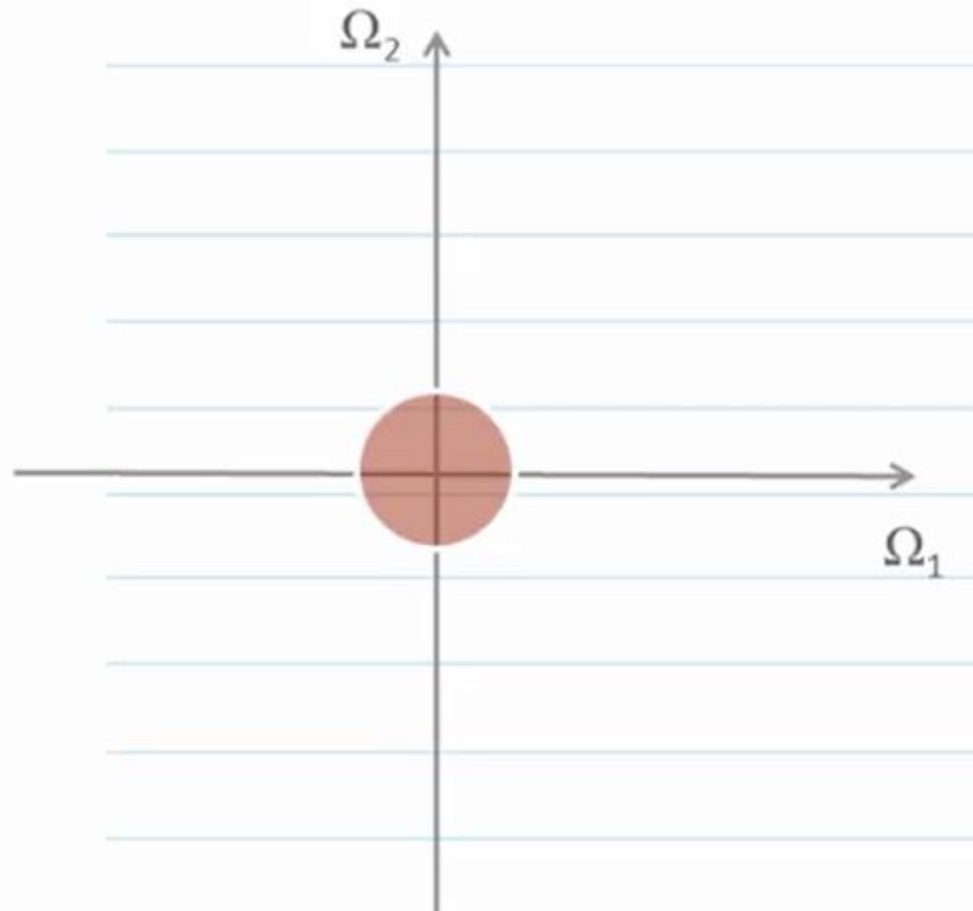
# Under Sampling



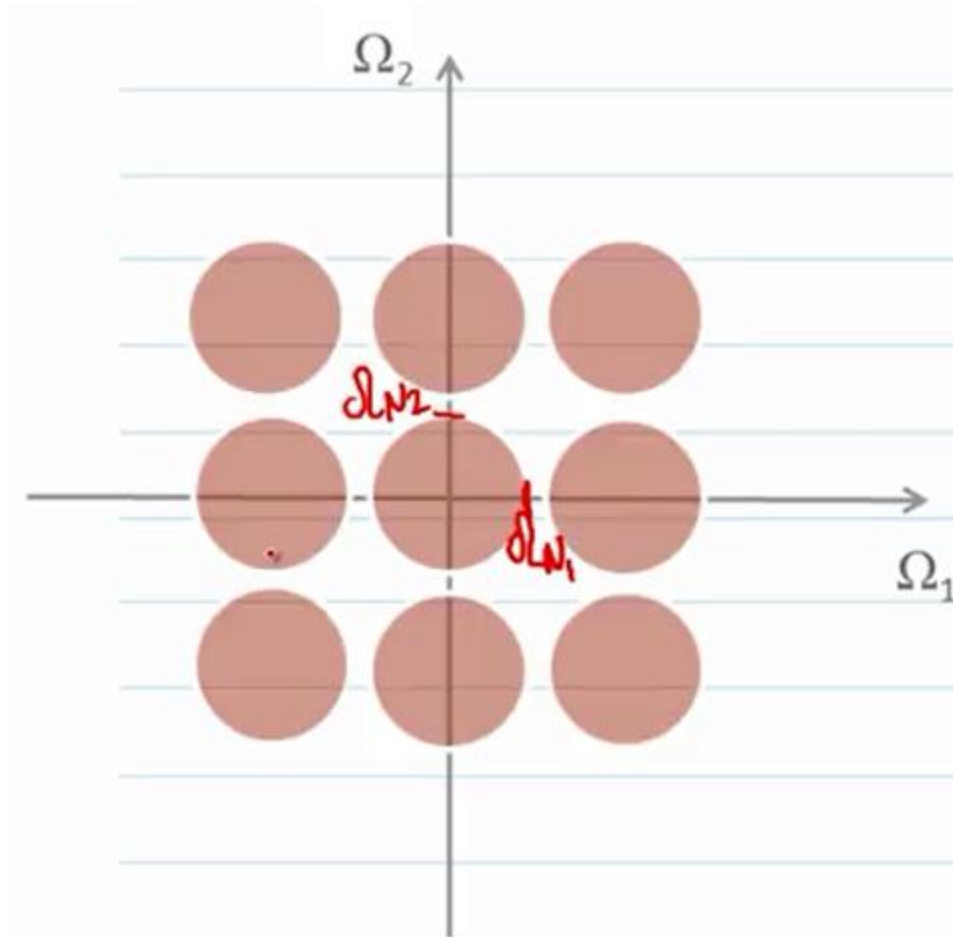
# Under Sampling



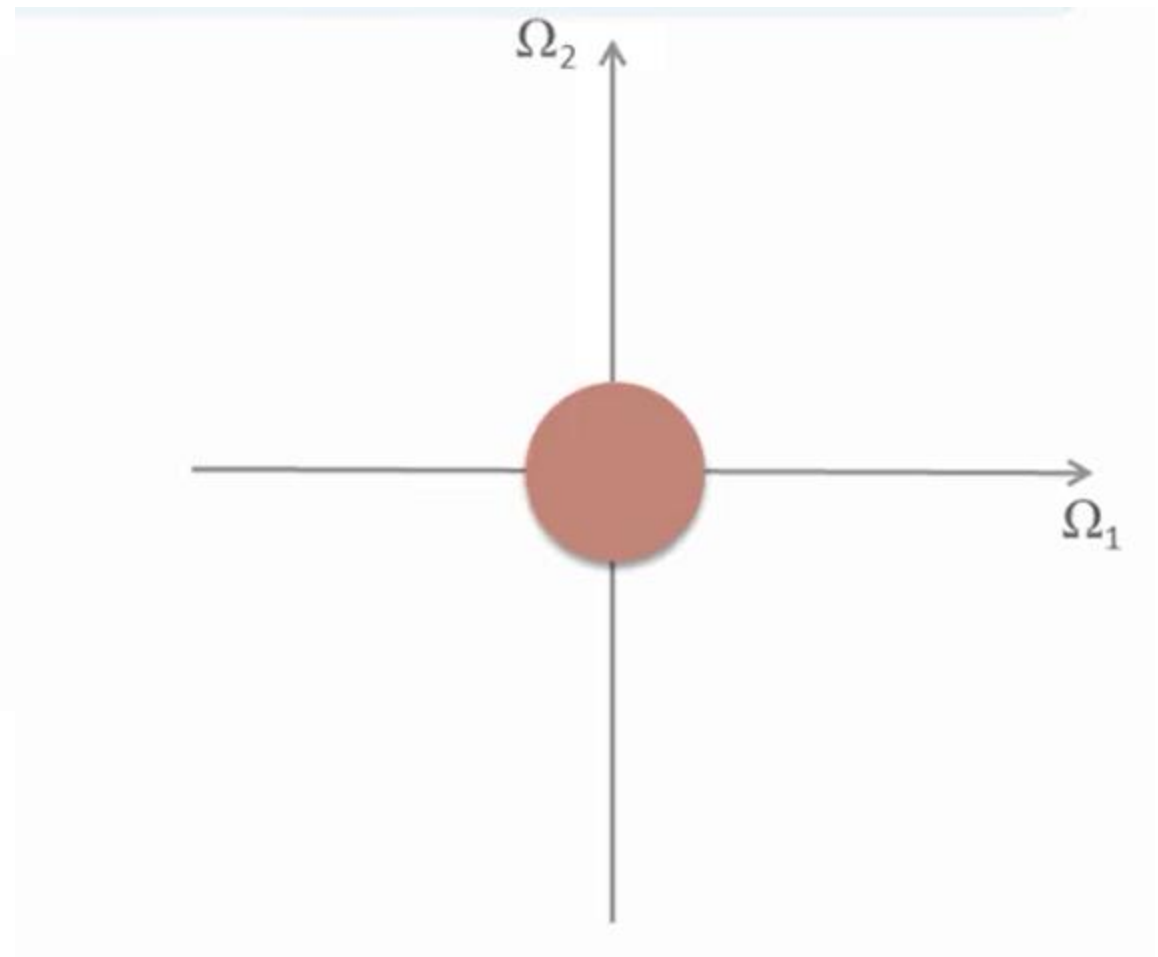
# 2D Nyquist Theorem



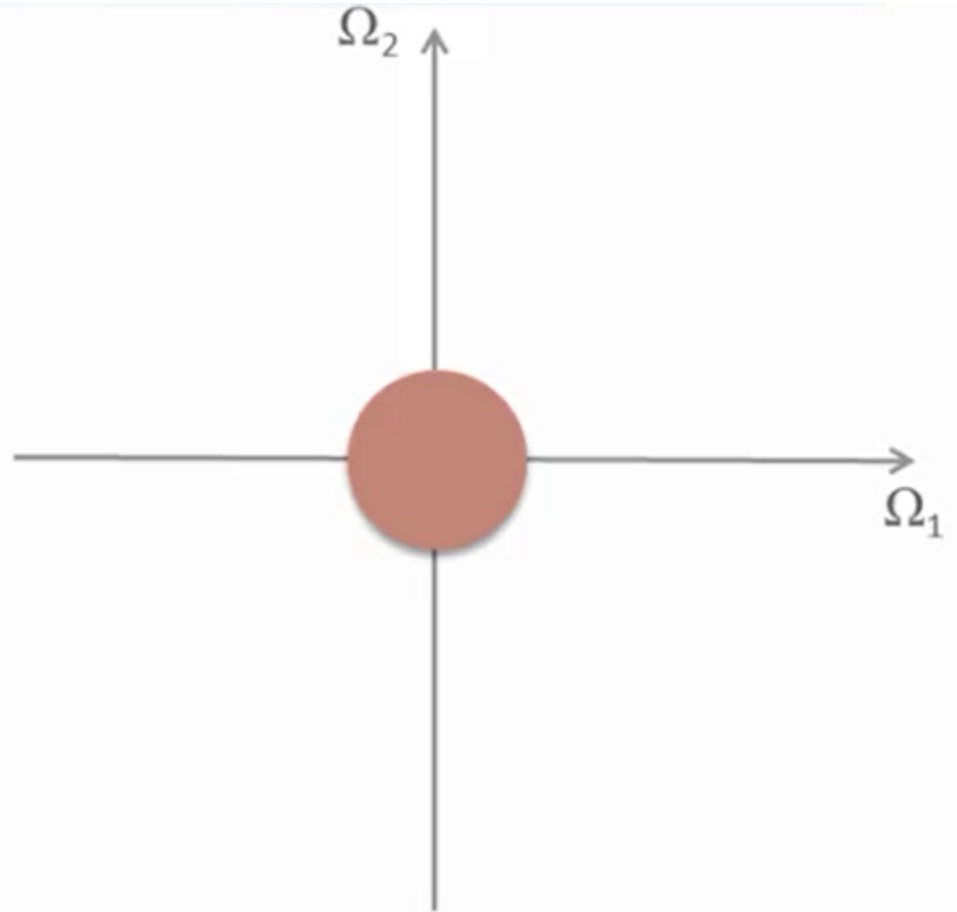
# 2D Nyquist Theorem



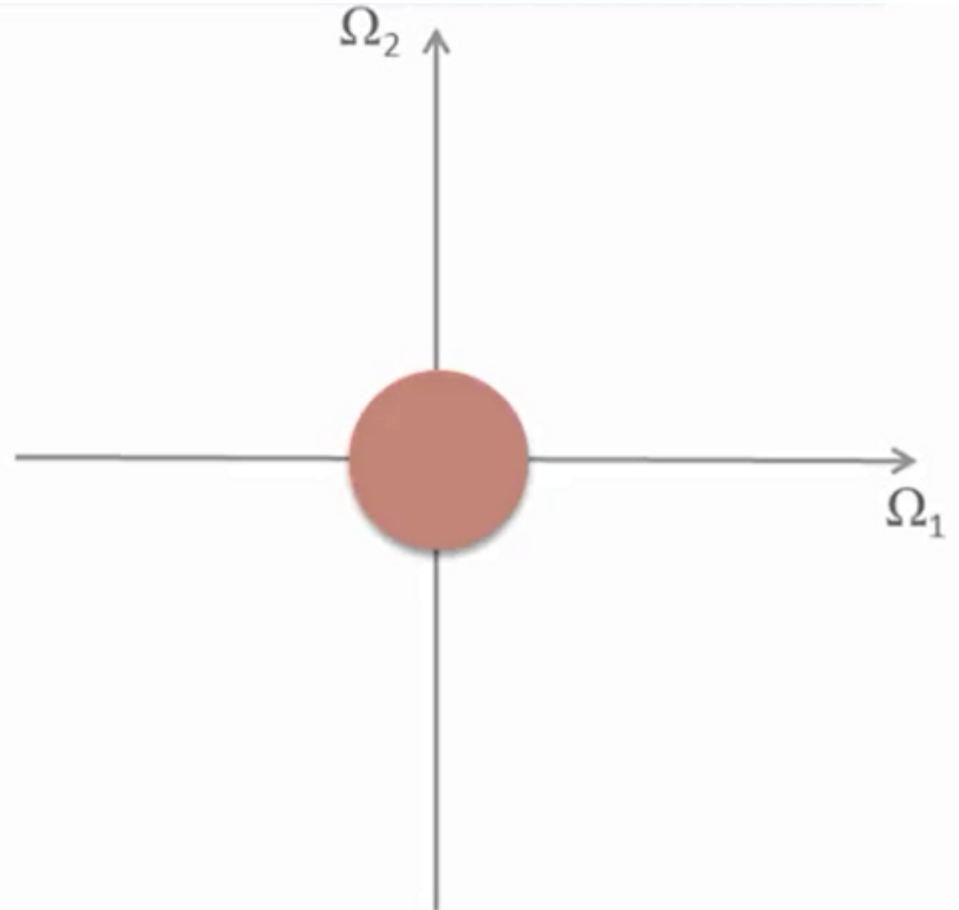
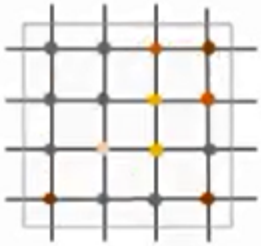
# Sampling in the Frequency



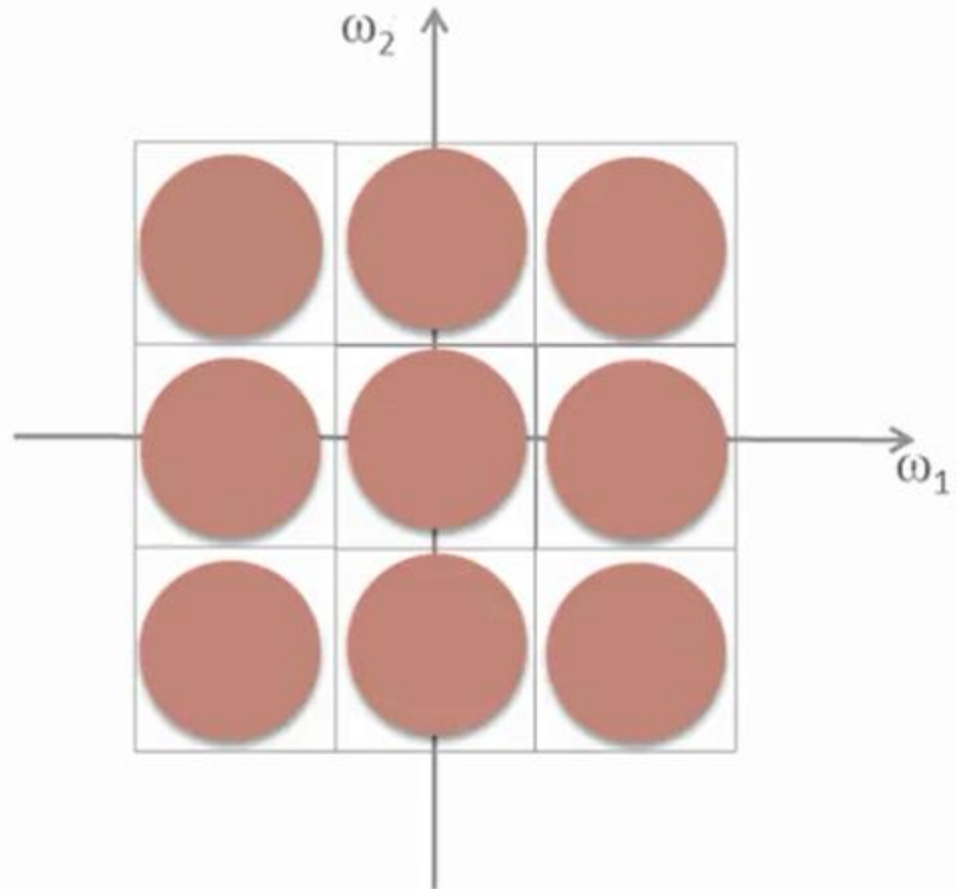
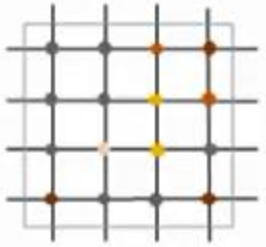
# Sampling in the Frequency



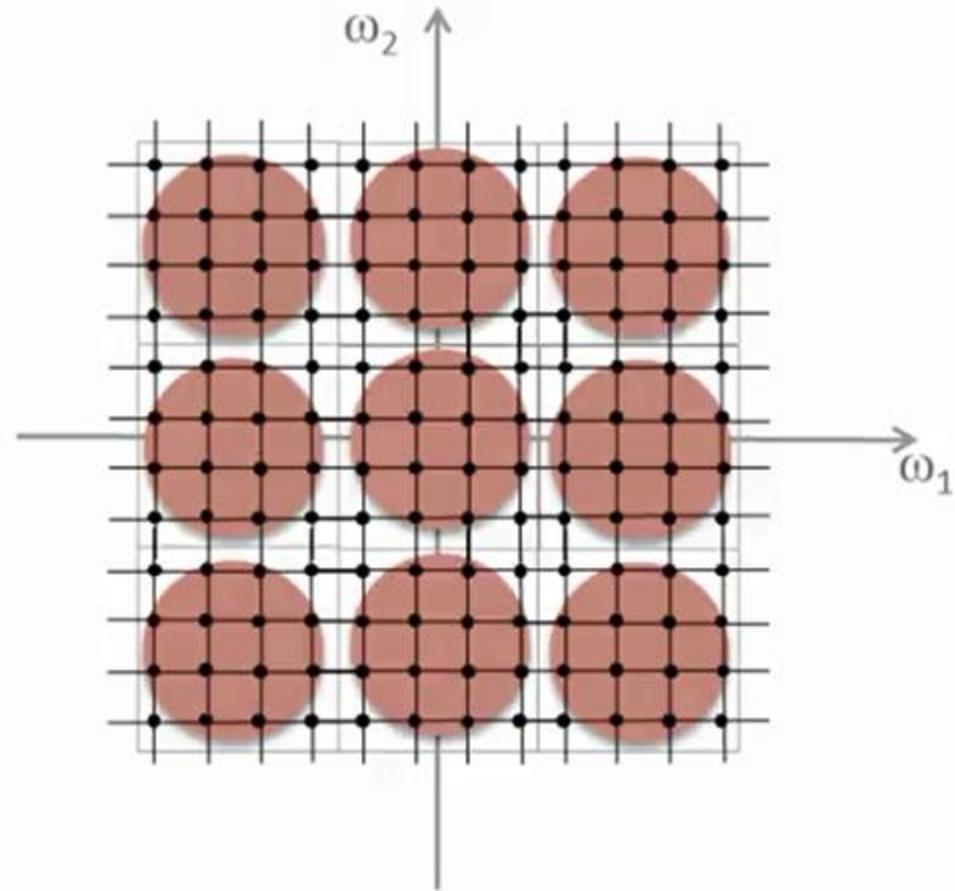
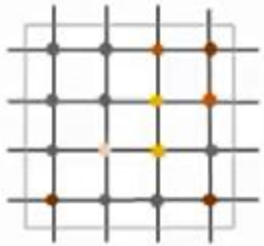
# Sampling in the Frequency



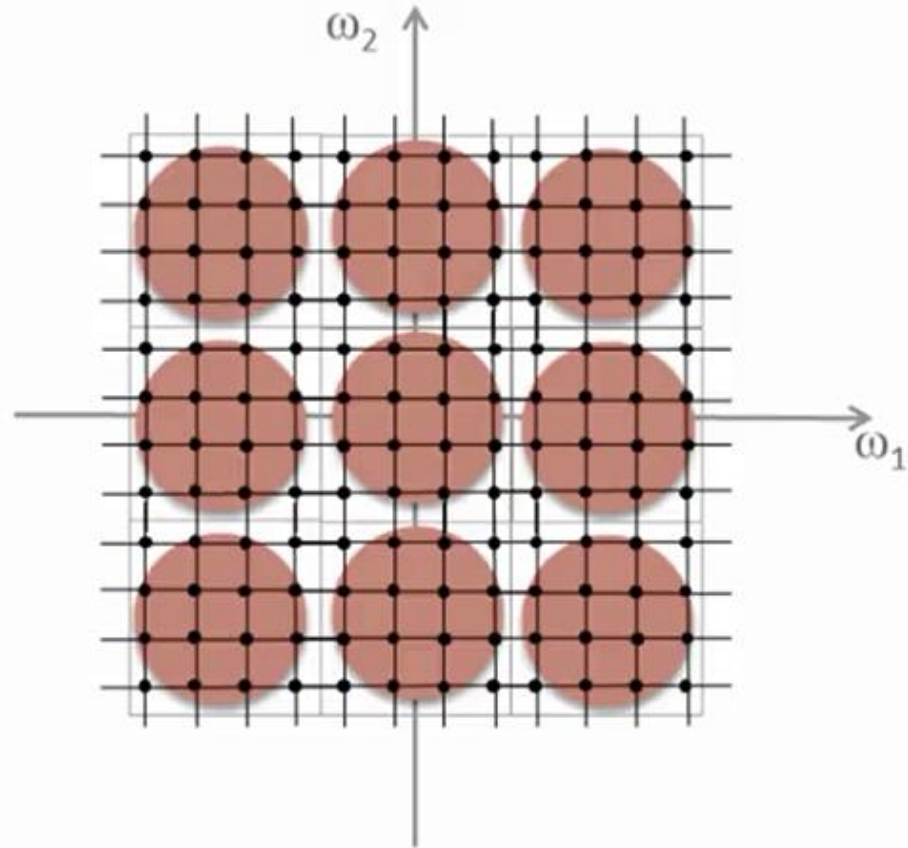
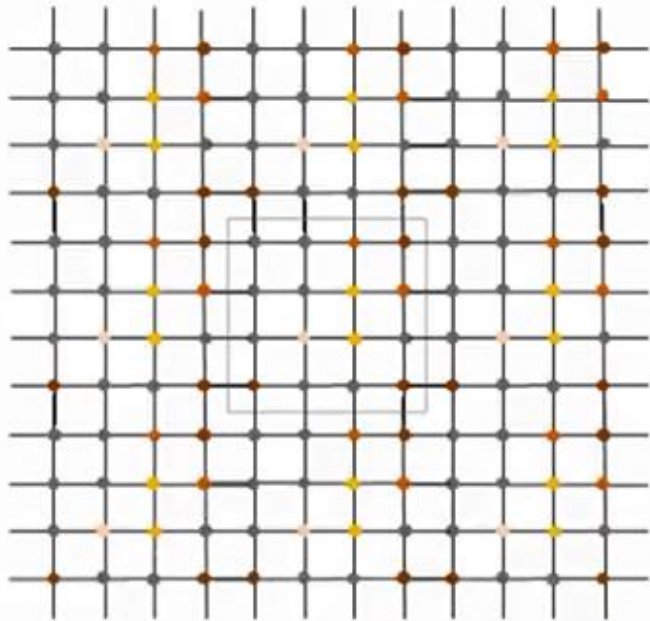
# Sampling in the Frequency



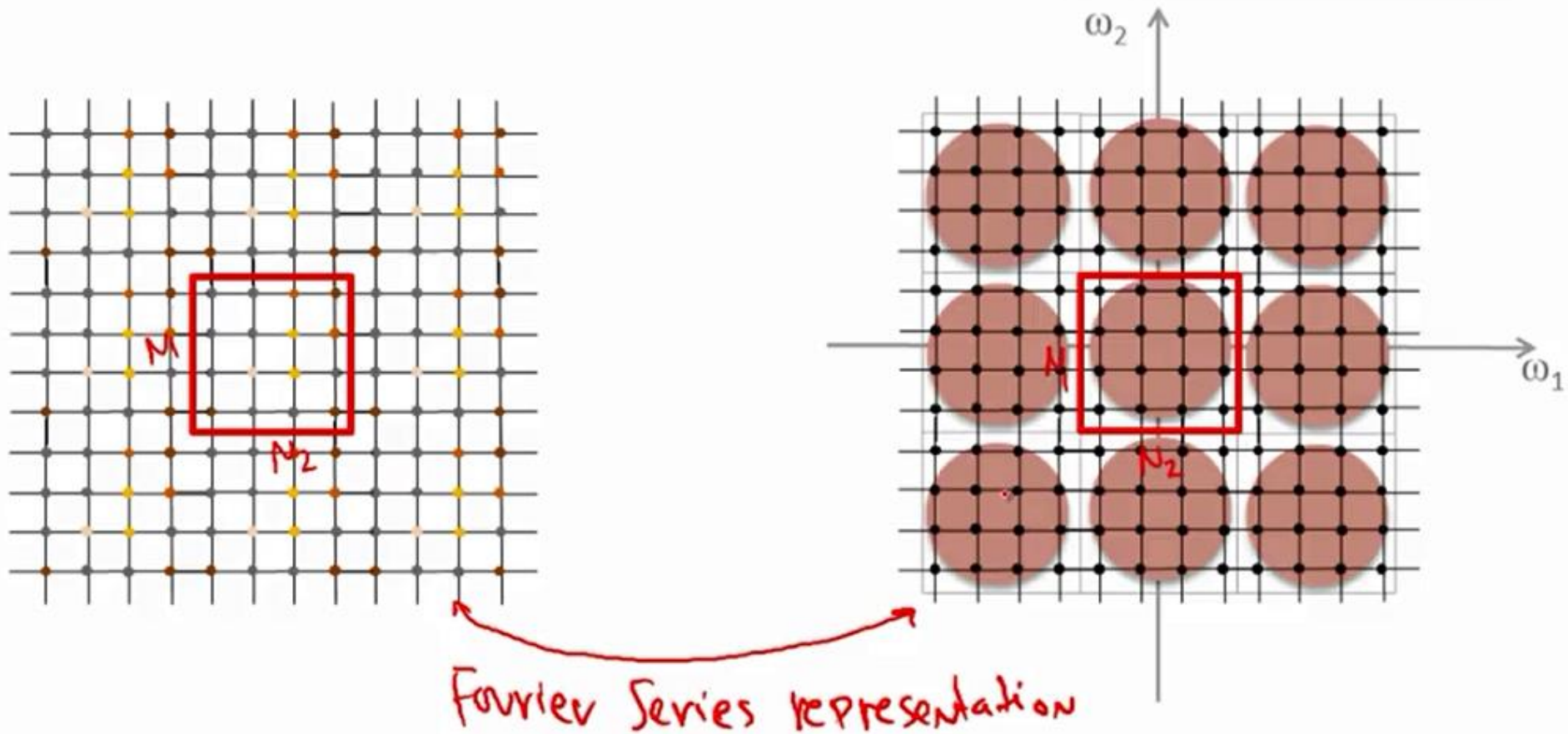
# Sampling in the Frequency



# Sampling in the Frequency



# Sampling in the Frequency



# 2D FT to 2D DFT

$$\underline{X(\omega_1, \omega_2)} = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \underline{x(n_1, n_2)} e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

$$X(k_1, k_2) = X(\omega_1, \omega_2) \Big|_{\omega_1 = \frac{2\pi}{N_1} k_1, \omega_2 = \frac{2\pi}{N_2} k_2} \quad \begin{cases} k_1 = 0, \dots, N_1 - 1 \\ k_2 = 0, \dots, N_2 - 1 \end{cases}$$

$$\underline{X(k_1, k_2)} = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) e^{-j \frac{2\pi}{N_1} n_1 k_1} e^{-j \frac{2\pi}{N_2} n_2 k_2} \quad \begin{cases} k_1 = 0, \dots, N_1 - 1 \\ k_2 = 0, \dots, N_2 - 1 \end{cases}$$

$$x(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X(k_1, k_2) e^{j \frac{2\pi}{N_1} n_1 k_1} e^{j \frac{2\pi}{N_2} n_2 k_2} \quad \begin{cases} n_1 = 0, \dots, N_1 - 1 \\ n_2 = 0, \dots, N_2 - 1 \end{cases}$$

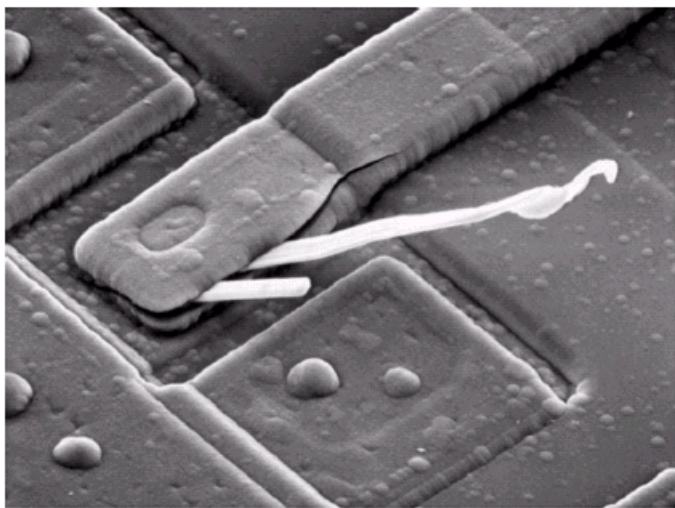
# The Discrete Fourier Transform (DFT)

The *Discrete Fourier Transform* of  $f(x, y)$ , for  $x = 0, 1, 2 \dots M-1$  and  $y = 0, 1, 2 \dots N-1$ , denoted by  $F(u, v)$ , is given by the equation:

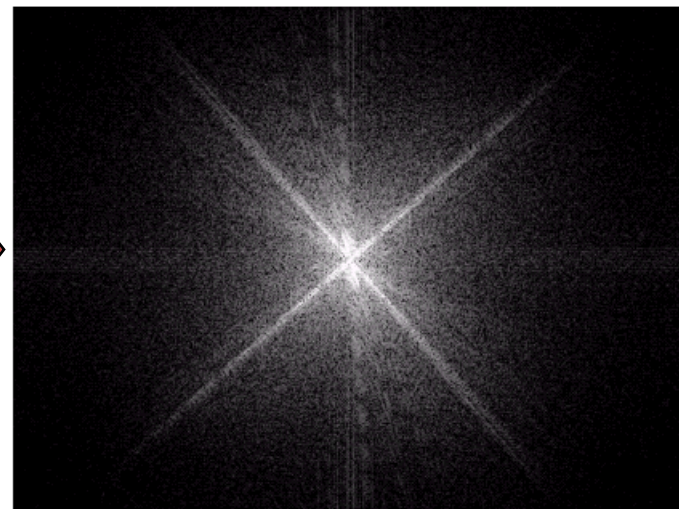
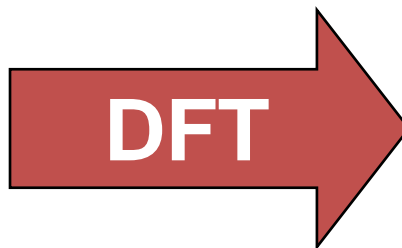
$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

for  $u = 0, 1, 2 \dots M-1$  and  $v = 0, 1, 2 \dots N-1$ .

# DFT & Images



Scanning electron microscope image of an integrated circuit magnified ~2500 times



Fourier spectrum of the image

# The Inverse DFT

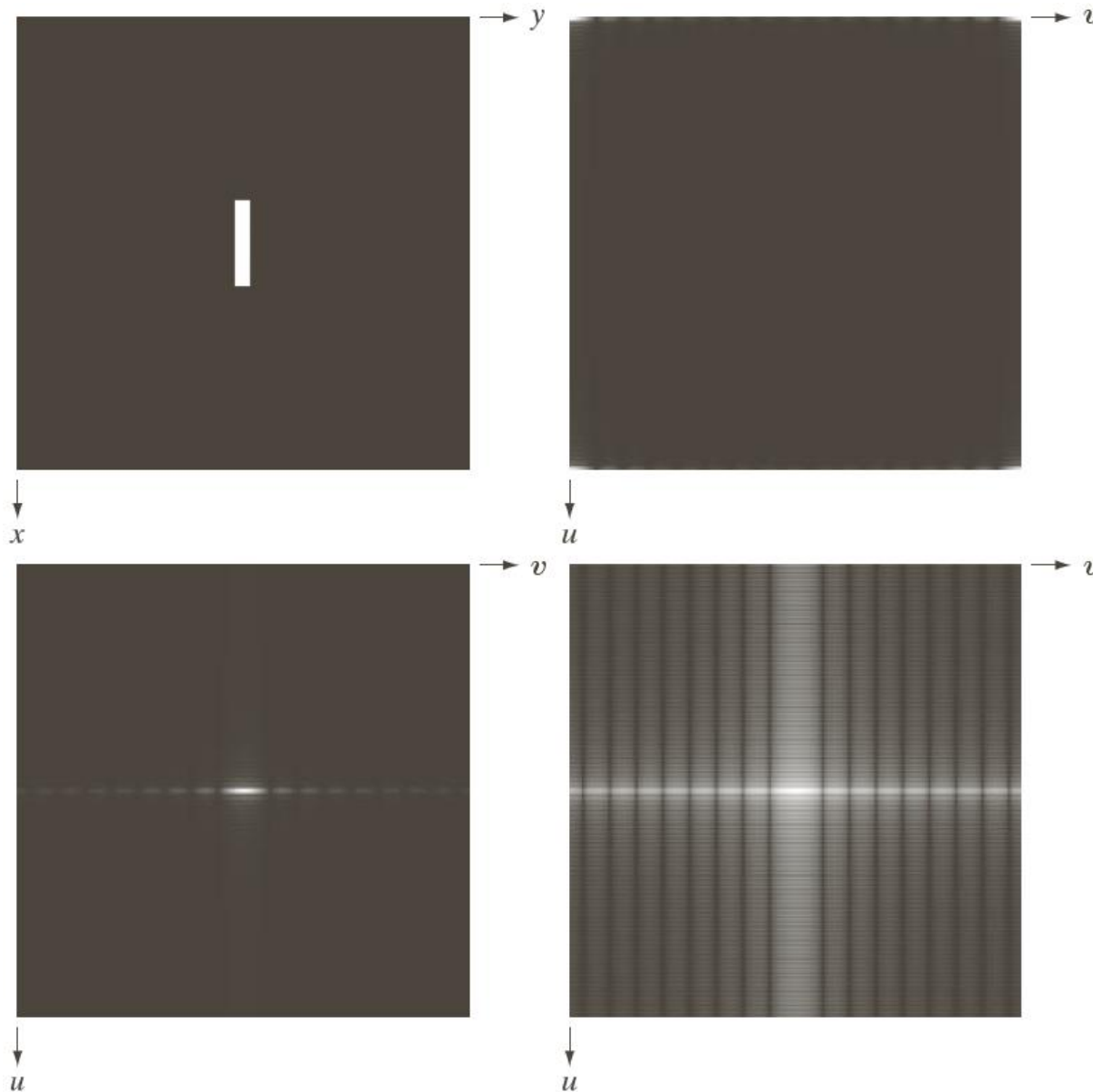
It is really important to note that the Fourier transform is completely **reversible**

The inverse DFT is given by:

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

for  $x = 0, 1, 2 \dots M-1$  and  $y = 0, 1, 2 \dots N-1$

# Frequencies in Images

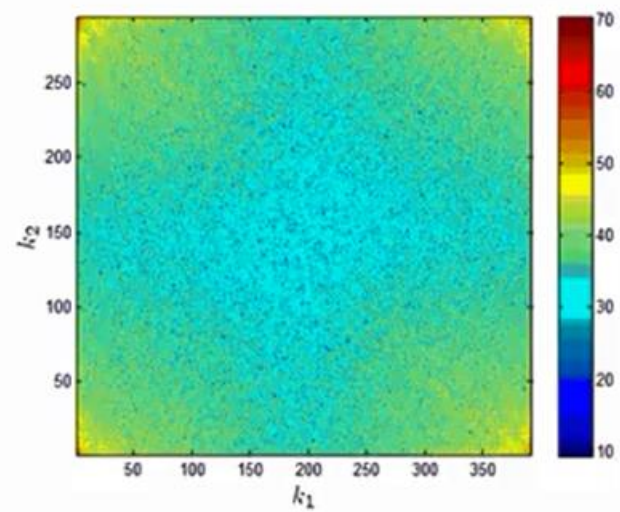
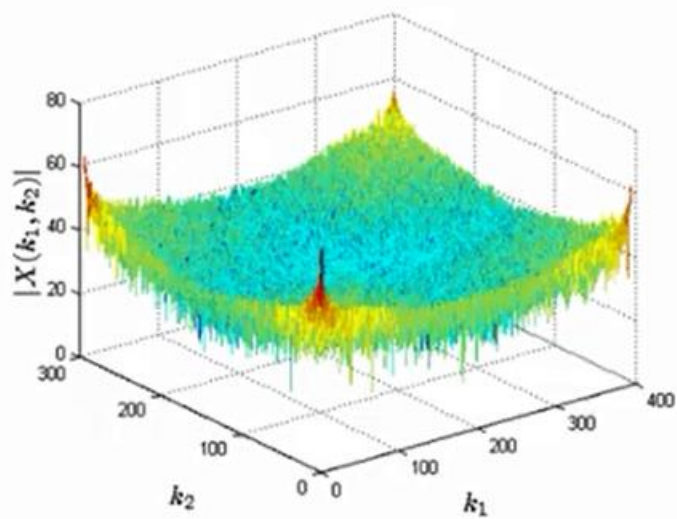


a	b
c	d

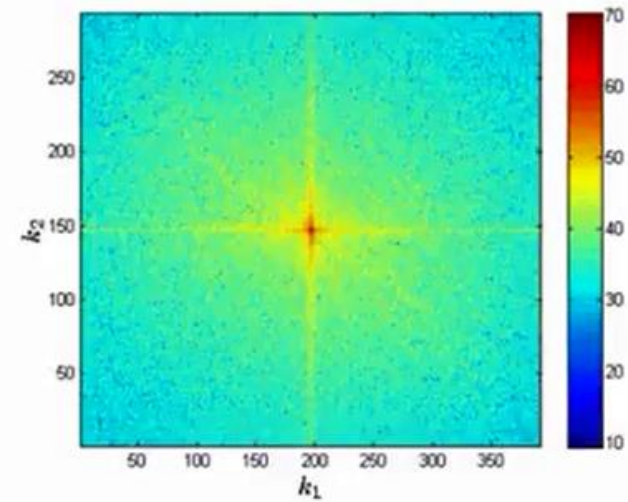
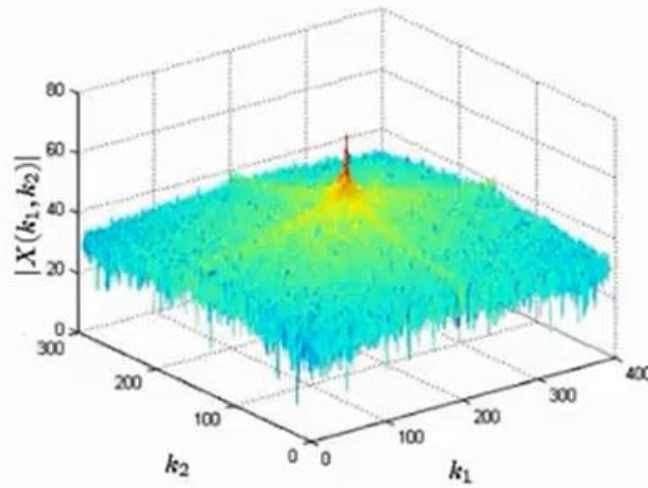
**FIGURE 4.24**

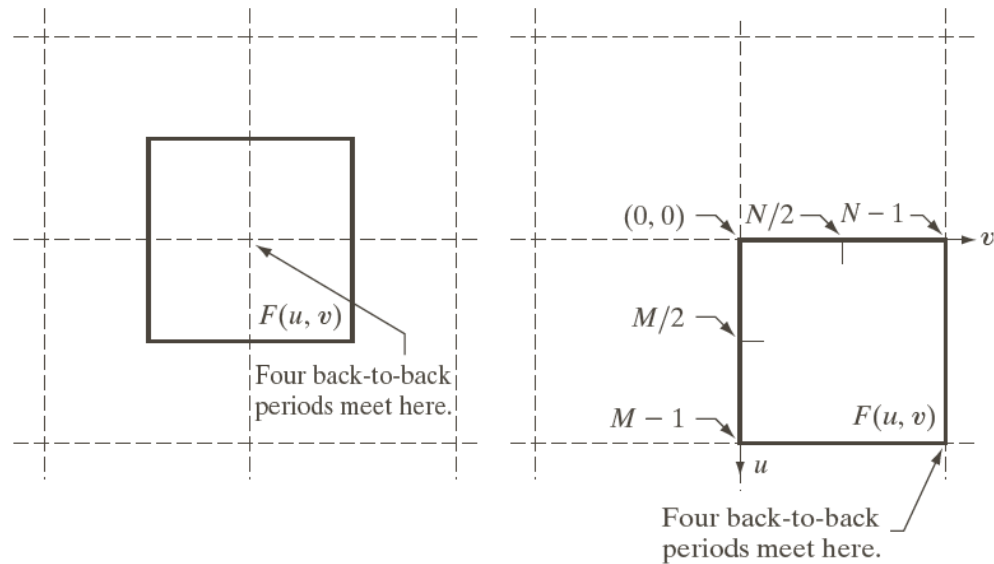
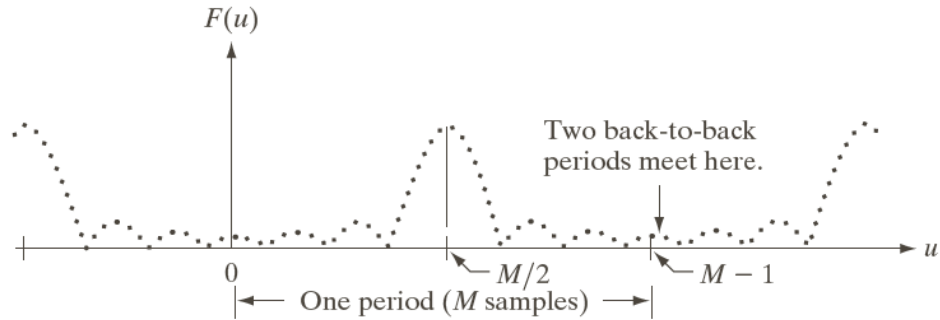
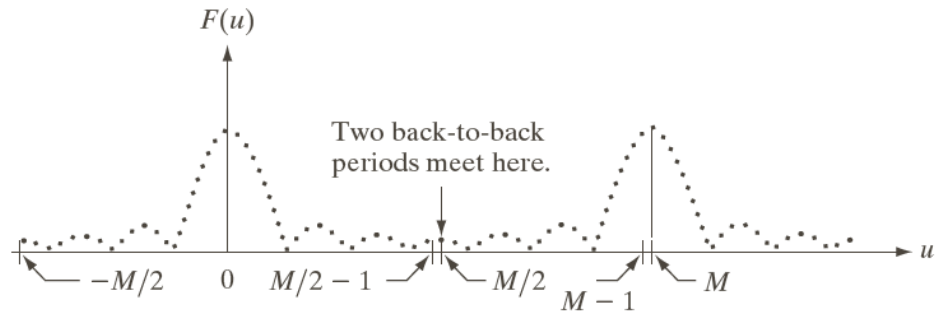
(a) Image.  
(b) Spectrum showing bright spots in the four corners.  
(c) Centered spectrum.  
(d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.

# DFT



# Centered DFT





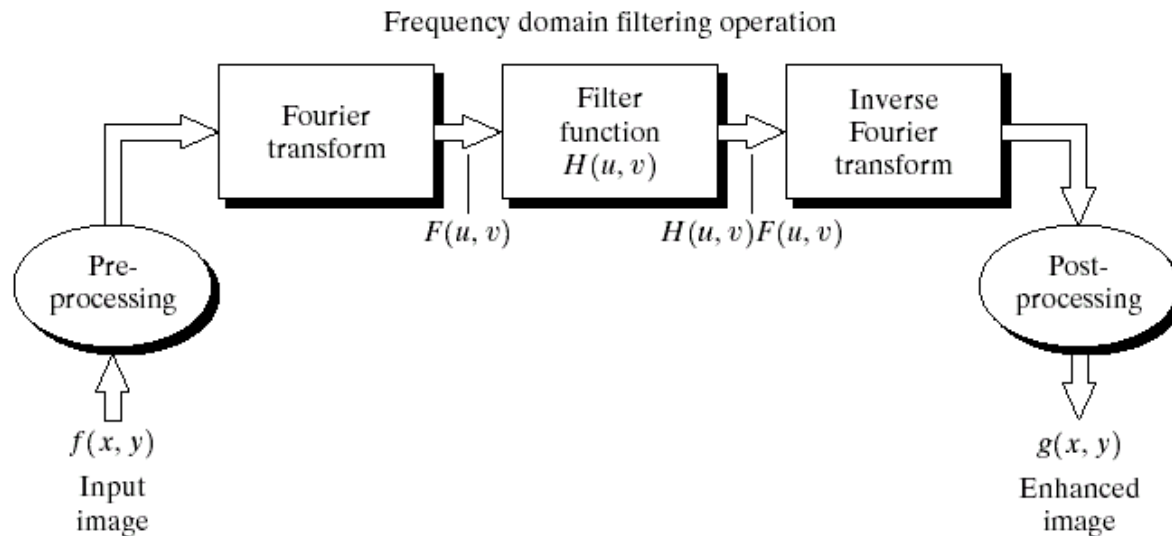
$\square$  = Periods of the DFT.

$\square$  =  $M \times N$  data array,  $F(u, v)$ .

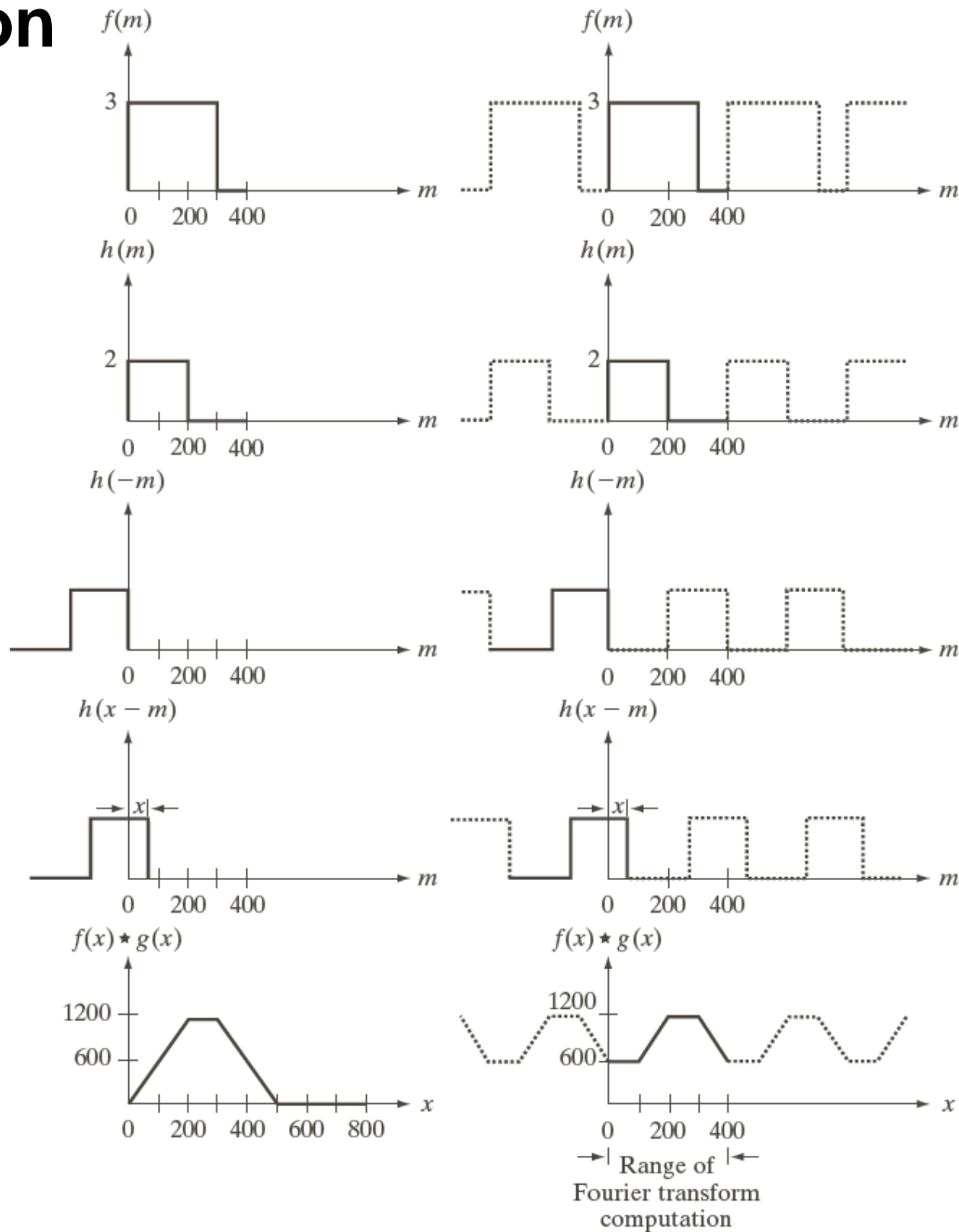
# The DFT and Image Processing

To filter an image in the frequency domain:

1. Compute  $F(u, v)$  the DFT of the image
2. Multiply  $F(u, v)$  by a filter function  $H(u, v)$
3. Compute the inverse DFT of the result



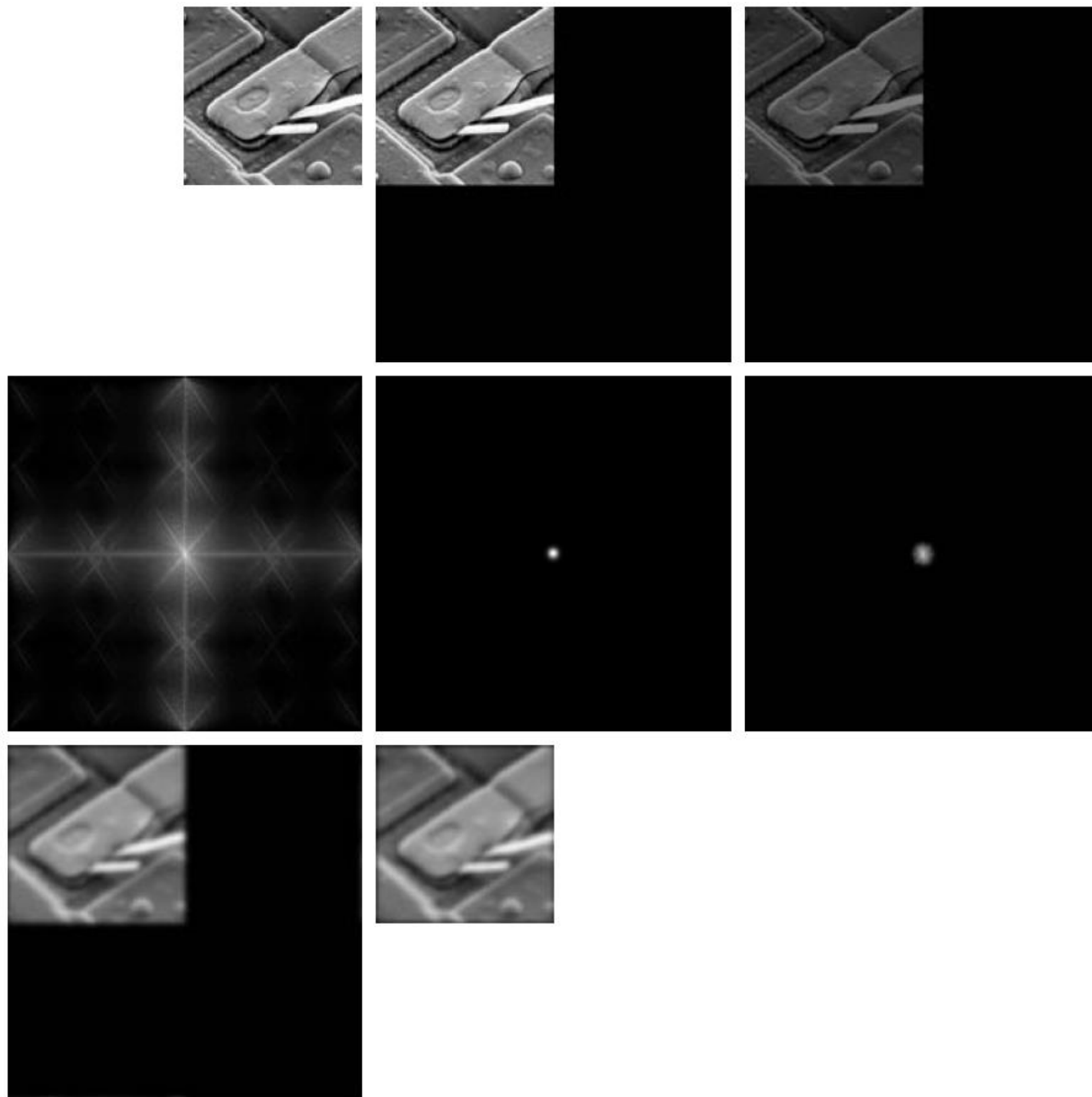
# Convolution



a	f
b	g
c	h
d	i
e	j

**FIGURE 4.28** Left column: convolution of two discrete functions obtained using the approach discussed in Section 3.4.2. The result in (e) is correct. Right column: Convolution of the same functions, but taking into account the periodicity implied by the DFT. Note in (j) how data from adjacent periods produce wraparound error, yielding an incorrect convolution result. To obtain the correct result, function padding must be used.

# Filtering In frequency Domain



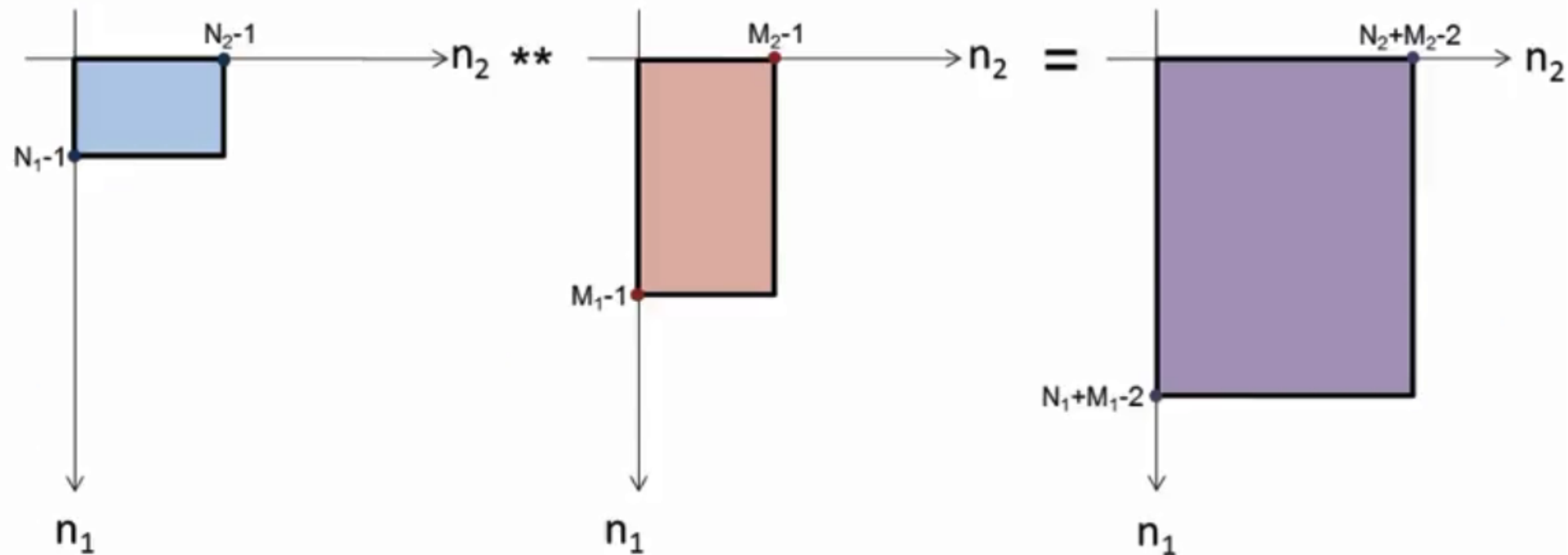
# 2D Circular Convolution

$x(n_1, n_2)$   $\longrightarrow$   $\boxed{\underline{h(n_1, n_2)}}$   $\longrightarrow$   $y_L(n_1, n_2)$  =  $x(n_1, n_2)$  \* \*  $h(n_1, n_2)$

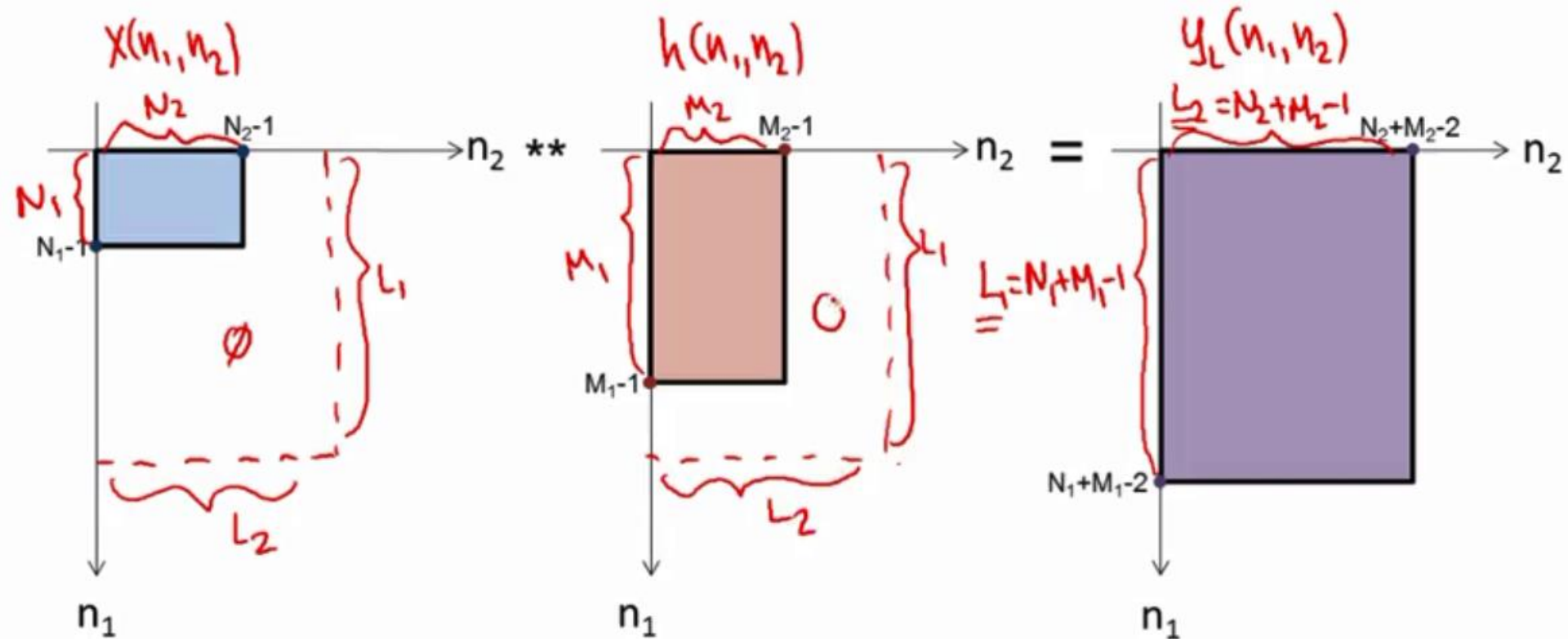
$X(k_1, k_2)$   $\longrightarrow$   $\boxed{\underline{H(k_1, k_2)}}$   $\longrightarrow$   $Y(k_1, k_2)$  =  $X(k_1, k_2)$   $\cdot$   $H(k_1, k_2)$

$y(n_1, n_2) = x(n_1, n_2) \underline{\circledast} \circledast h(n_1, n_2)$

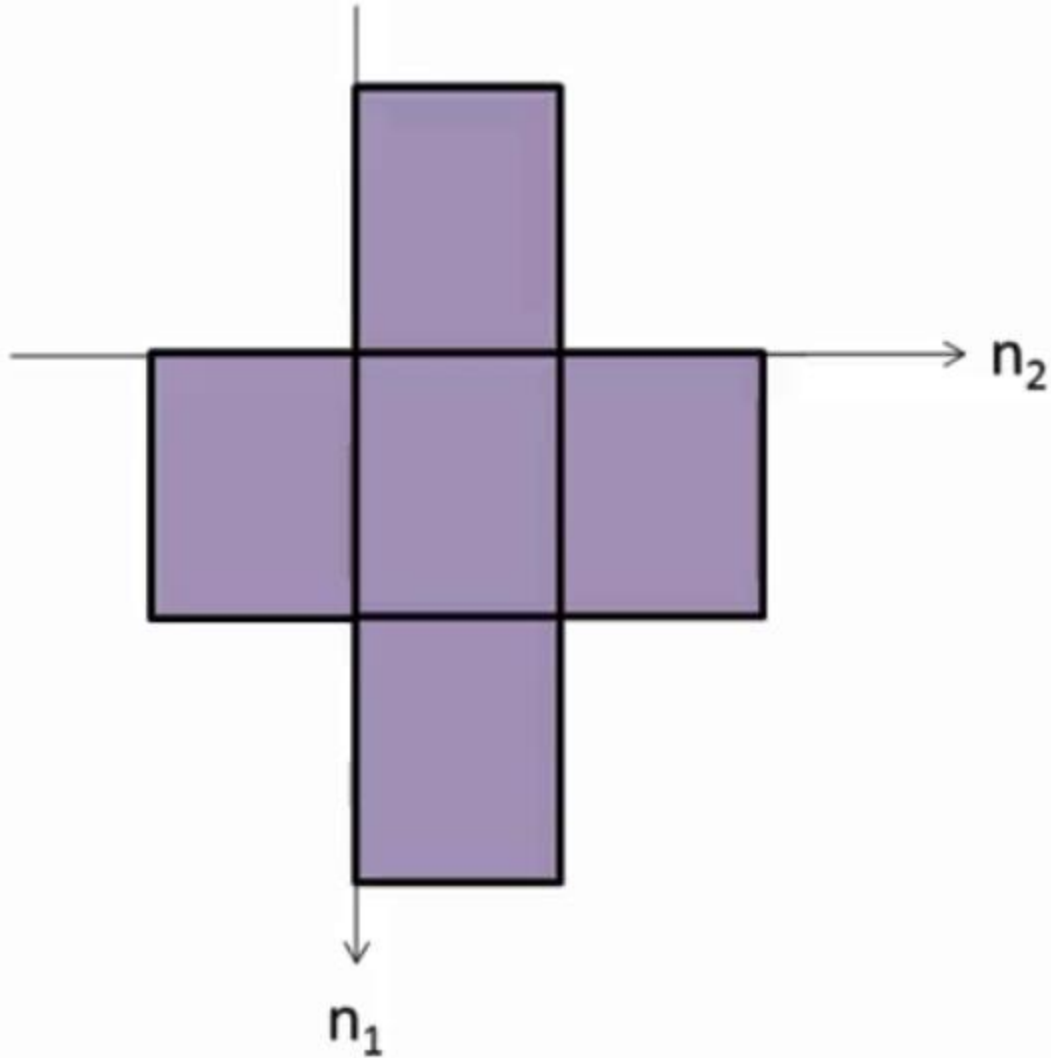
# 2D Linear Convolution



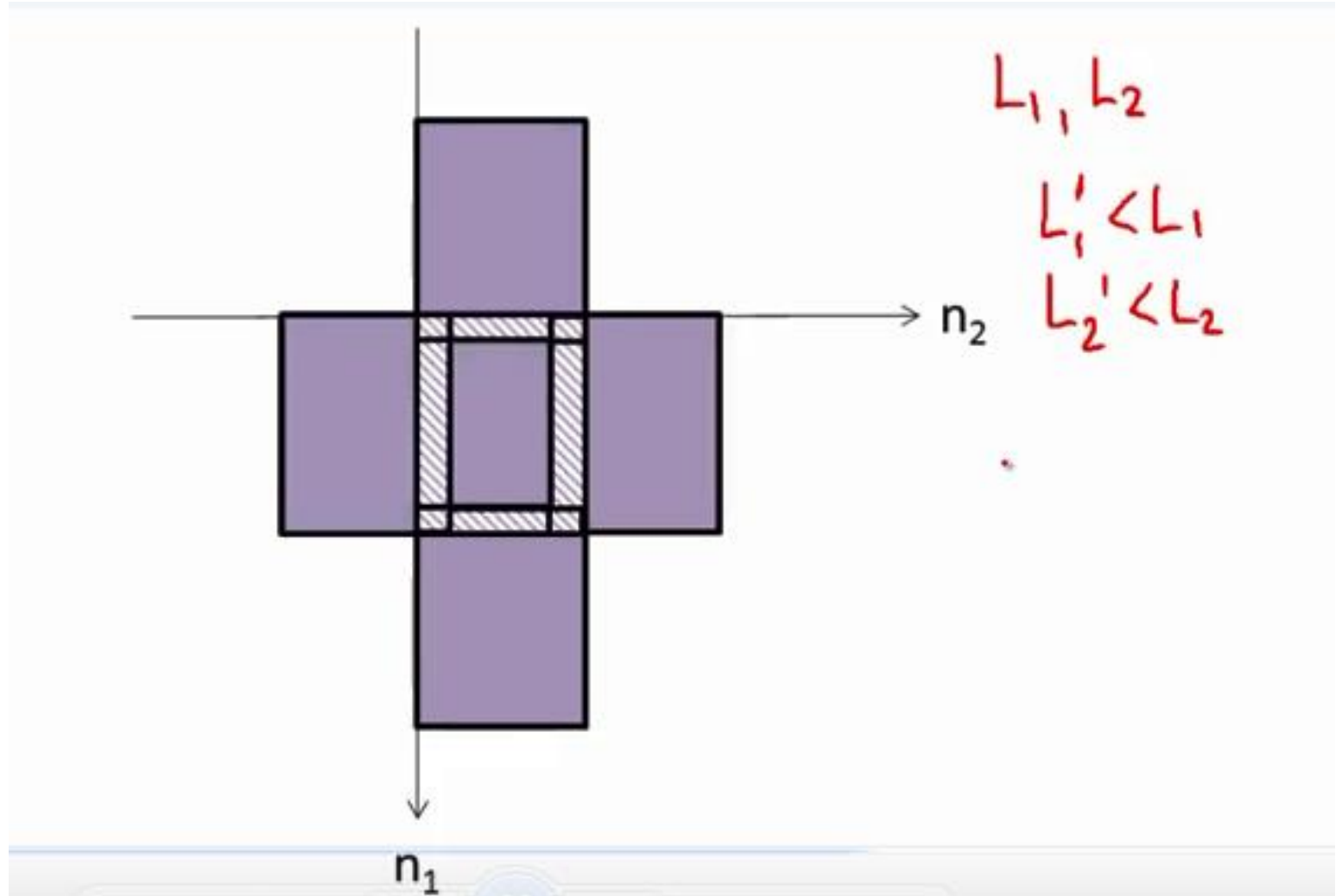
# 2D Linear & Circular Convolution

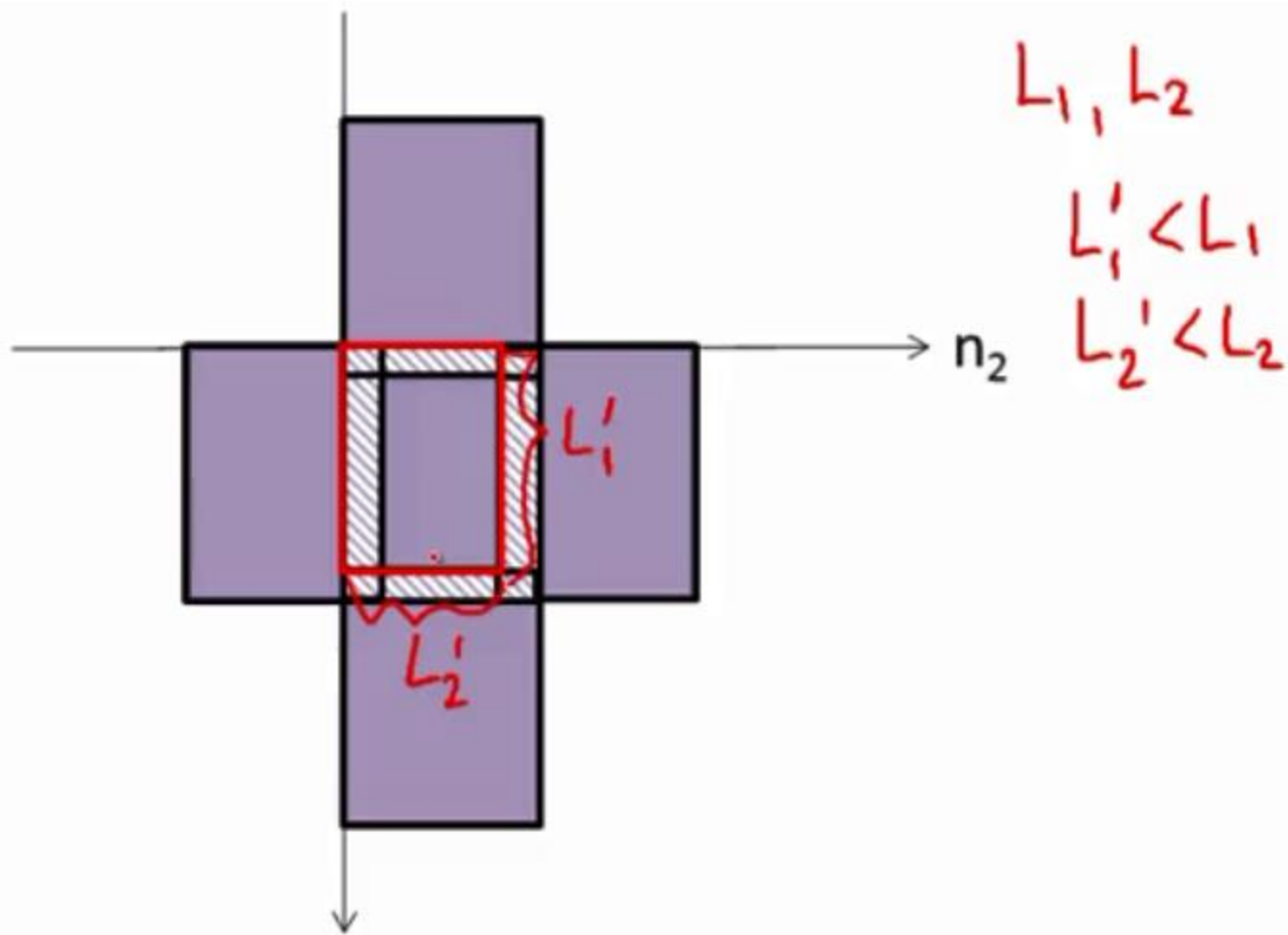


# Spatial Aliasing

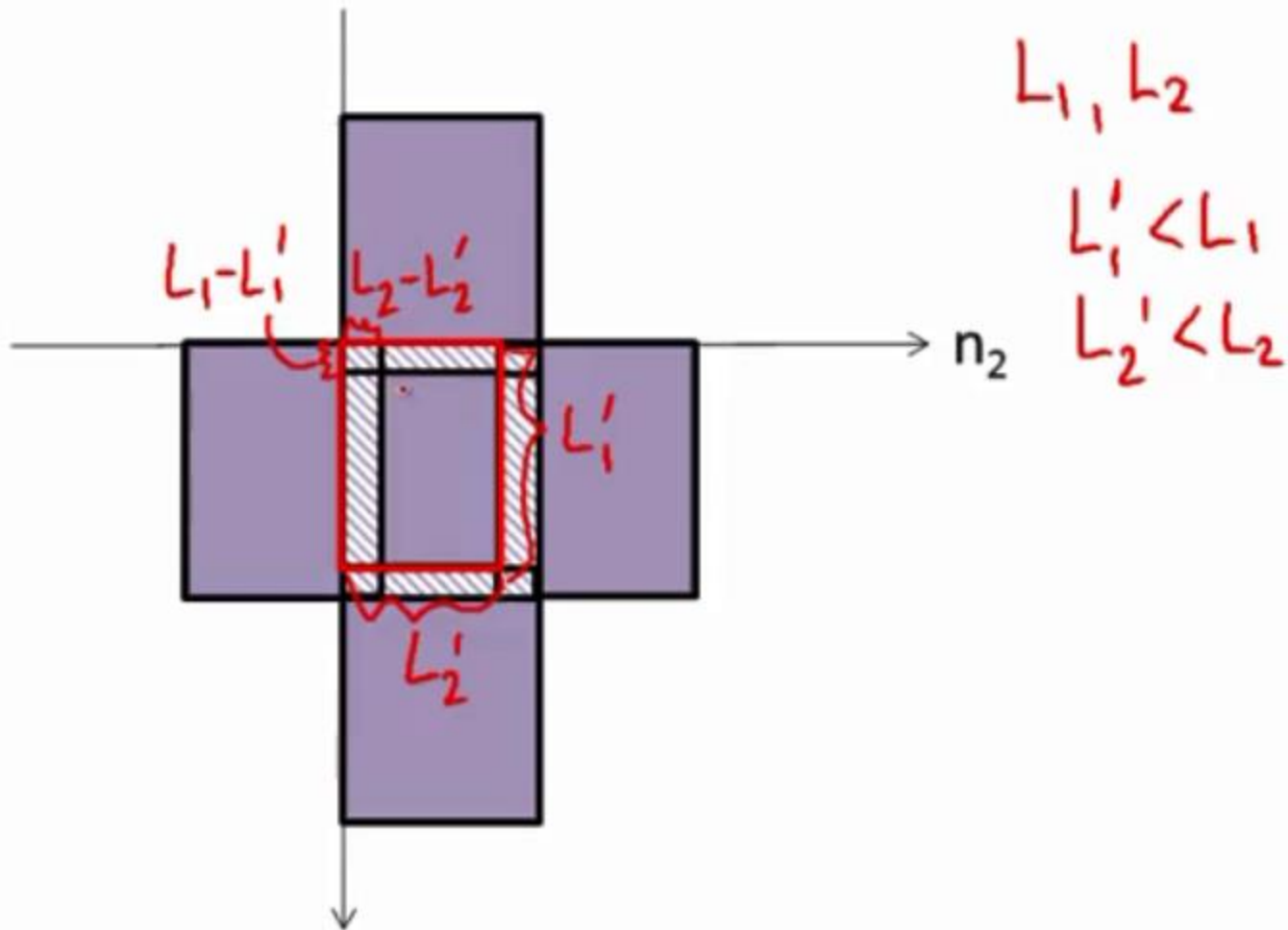


# Spatial Aliasing

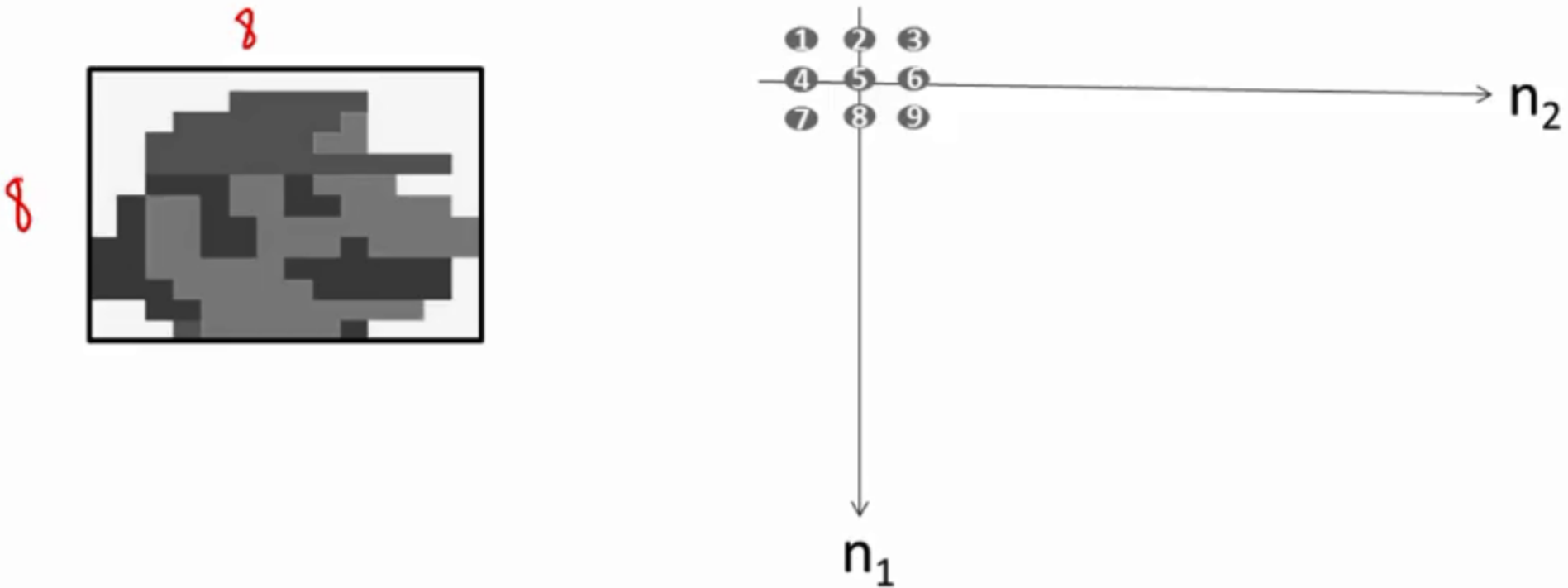




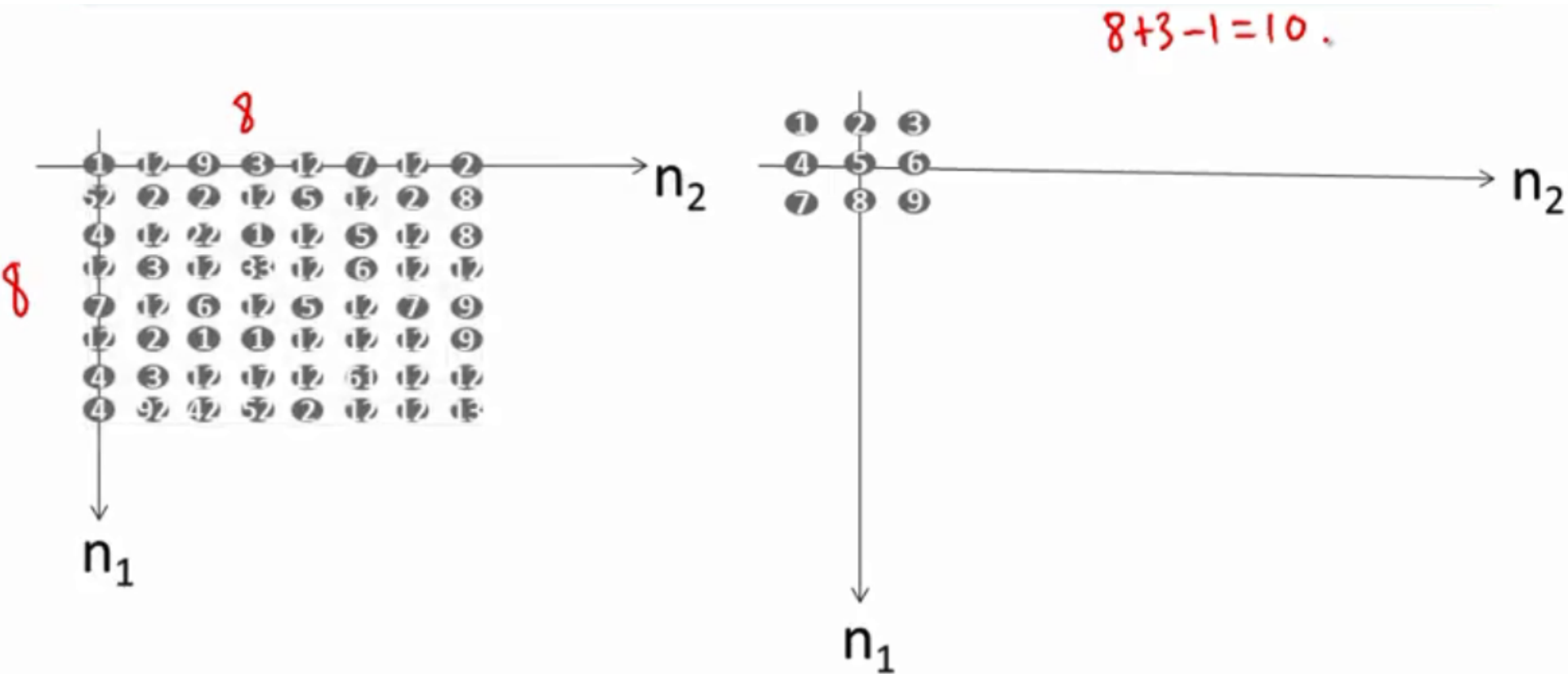
# Spatial Aliasing



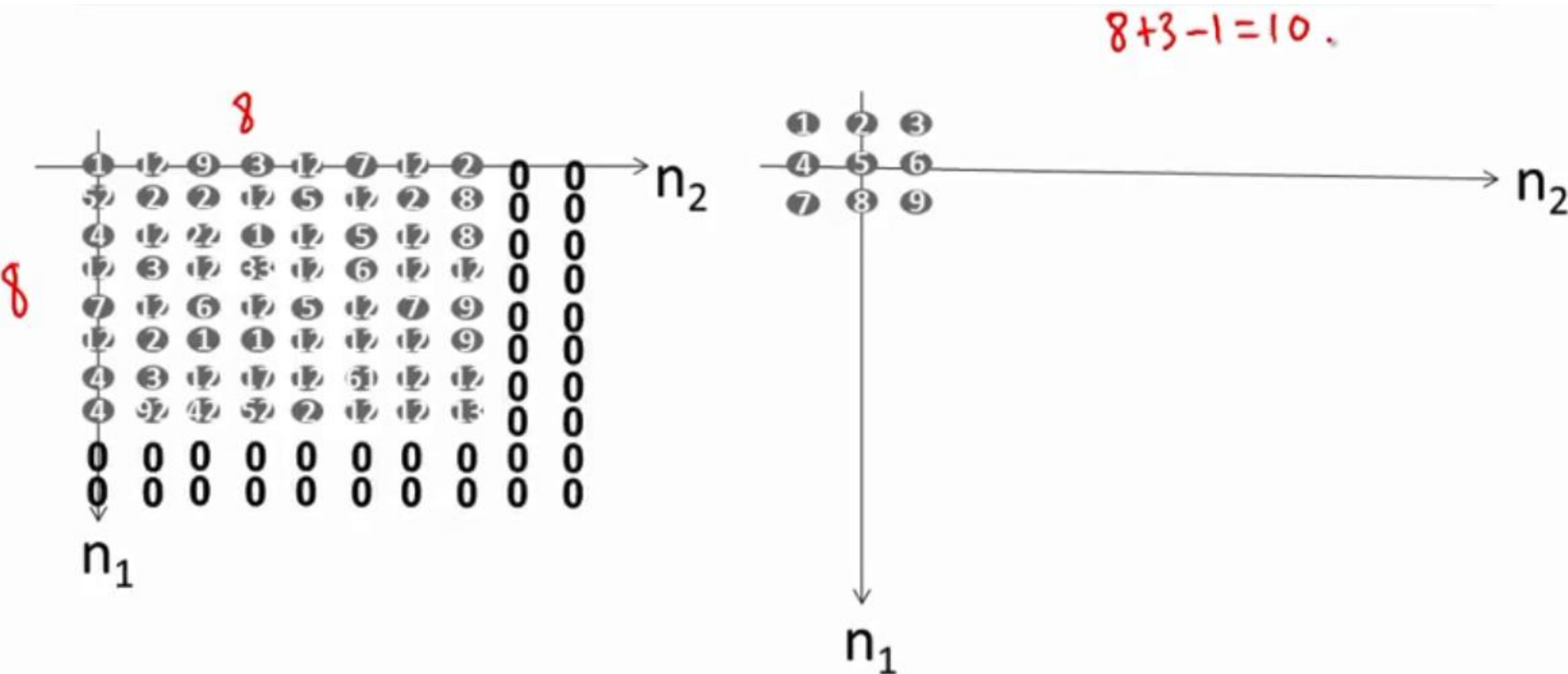
# DFT for Filtering



# DFT for Filtering

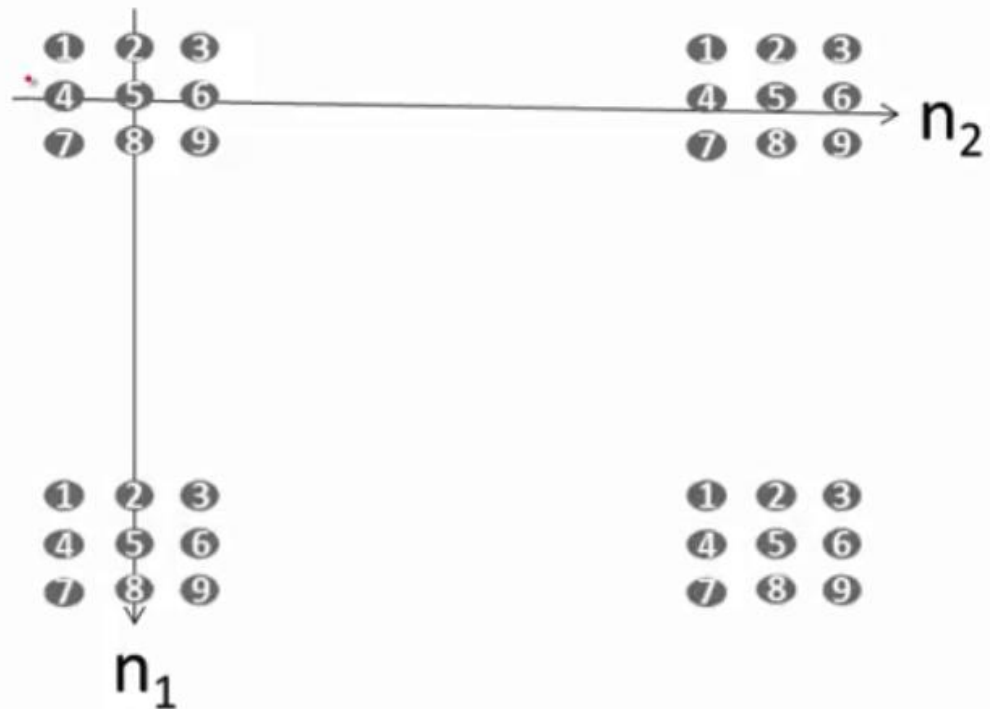
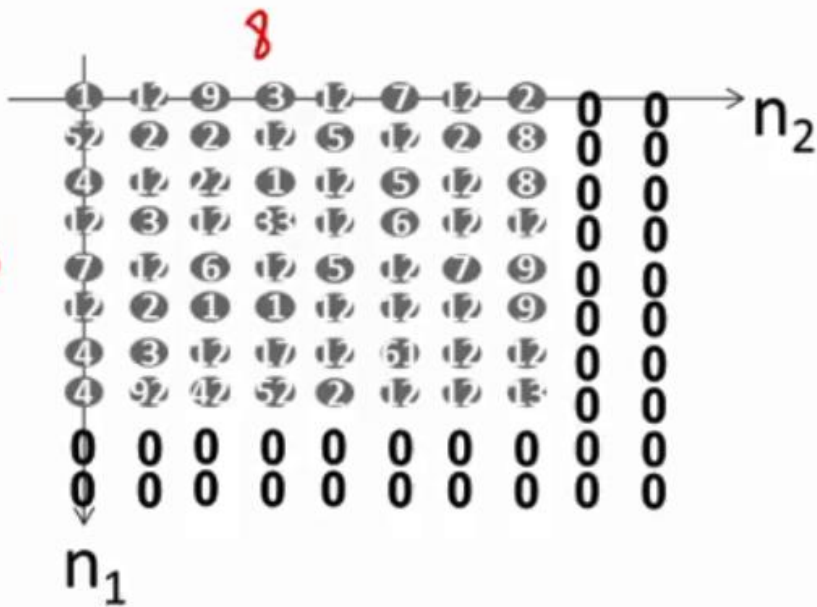


# DFT for Filtering

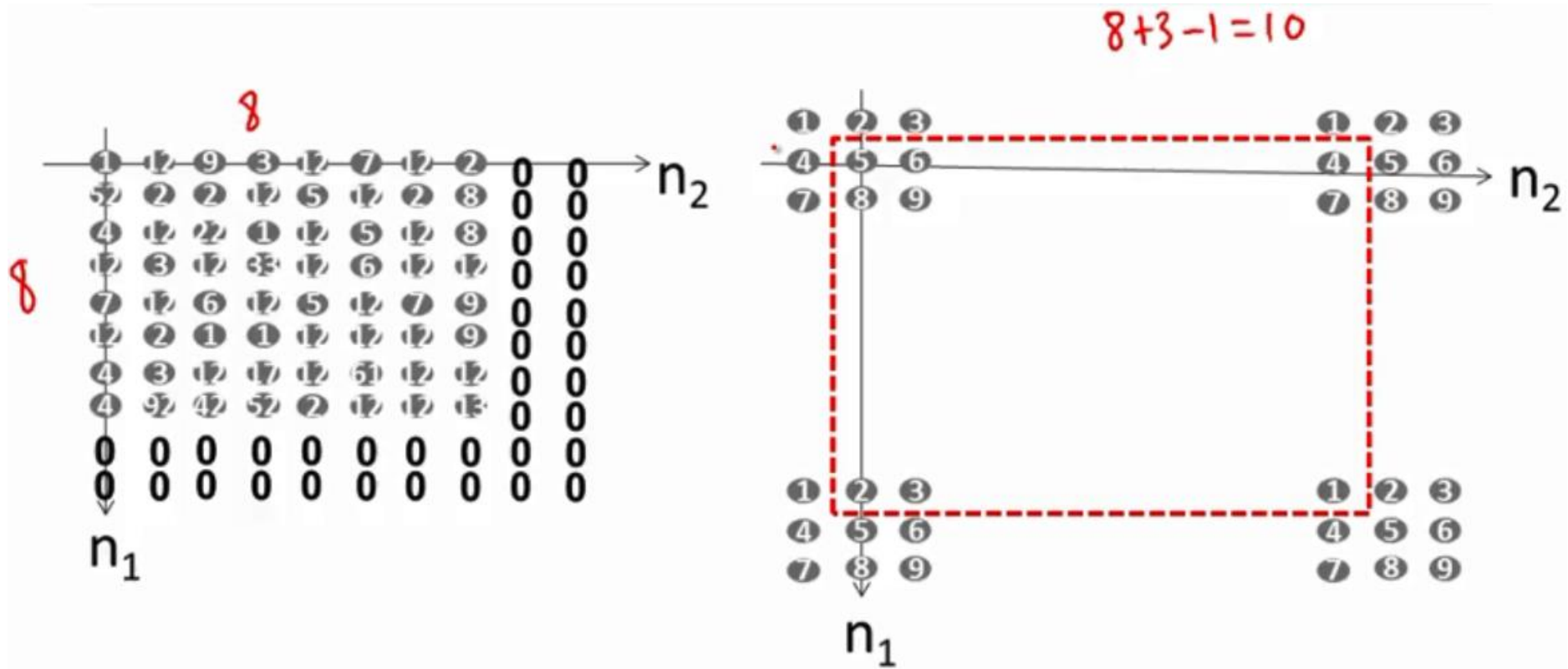


# DFT for Filtering

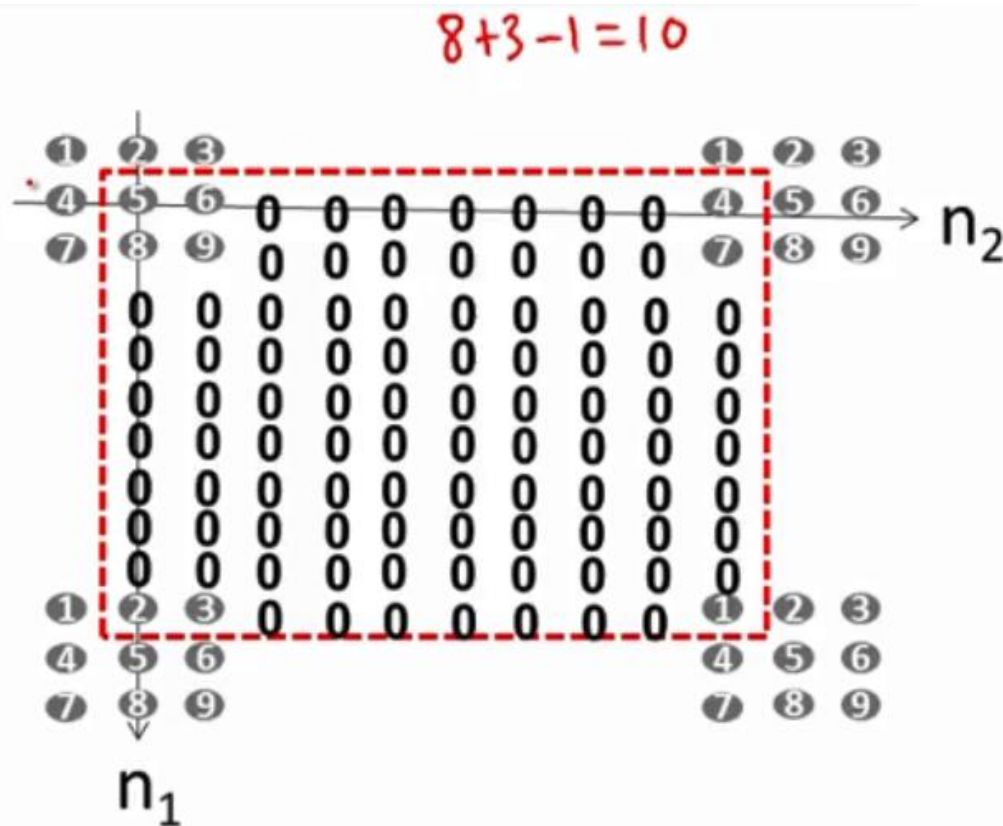
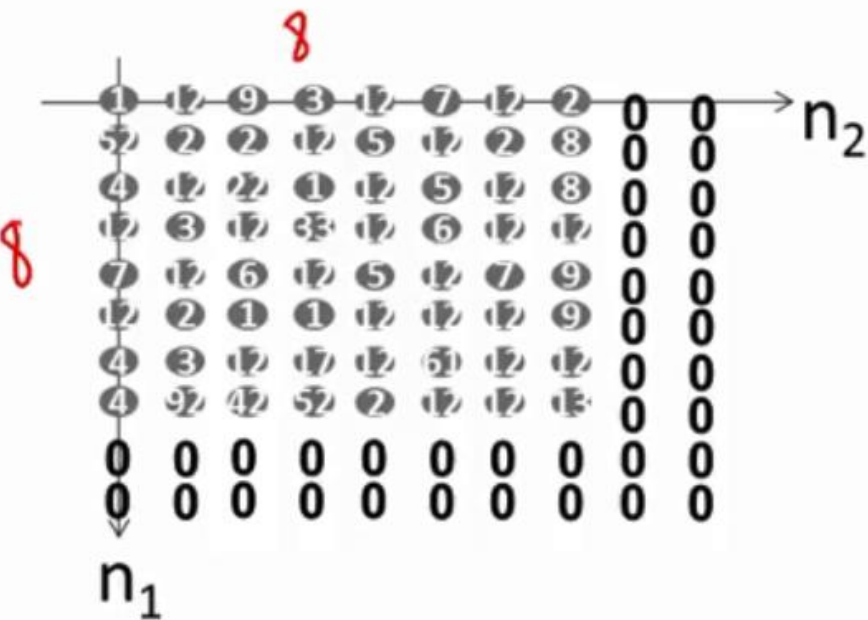
$$8+3-1=10$$



# DFT for Filtering



# DFT for Filtering



# Readings from Book (3<sup>rd</sup> Edn.)

- Frequency Domain (Chapter-4)



# Acknowledgements

- ◆ Digital Image Processing”, Rafael C. Gonzalez & Richard E. Woods, Addison-Wesley, 2002
- ◆ Brian Mac Namee, Digital Image Processing, School of Computing, Dublin Institute of Technology
- ◆ Digital Image processing Lectures: Coursera

Material in these slides has been taken from, the following resources