Digital Image Processing

Lecture # 12
Frequency Domain Image Analysis

Image Enhancement in Frequency Domain

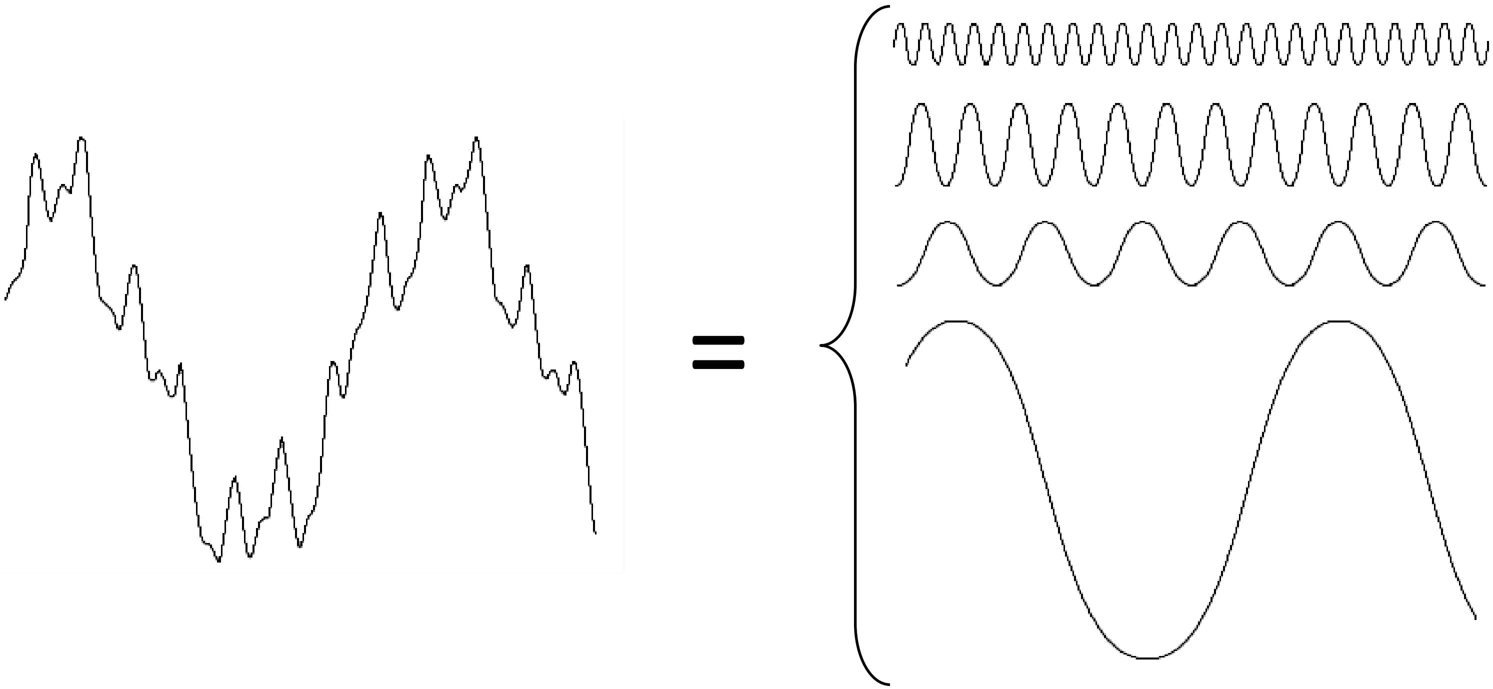
Joseph Fourier (1768 – 1830)

– Most famous for his work *"La Théorie Analitique de la Chaleur"* published in 1822
– Translated into English in 1878: *"The Analytic Theory of Heat"*

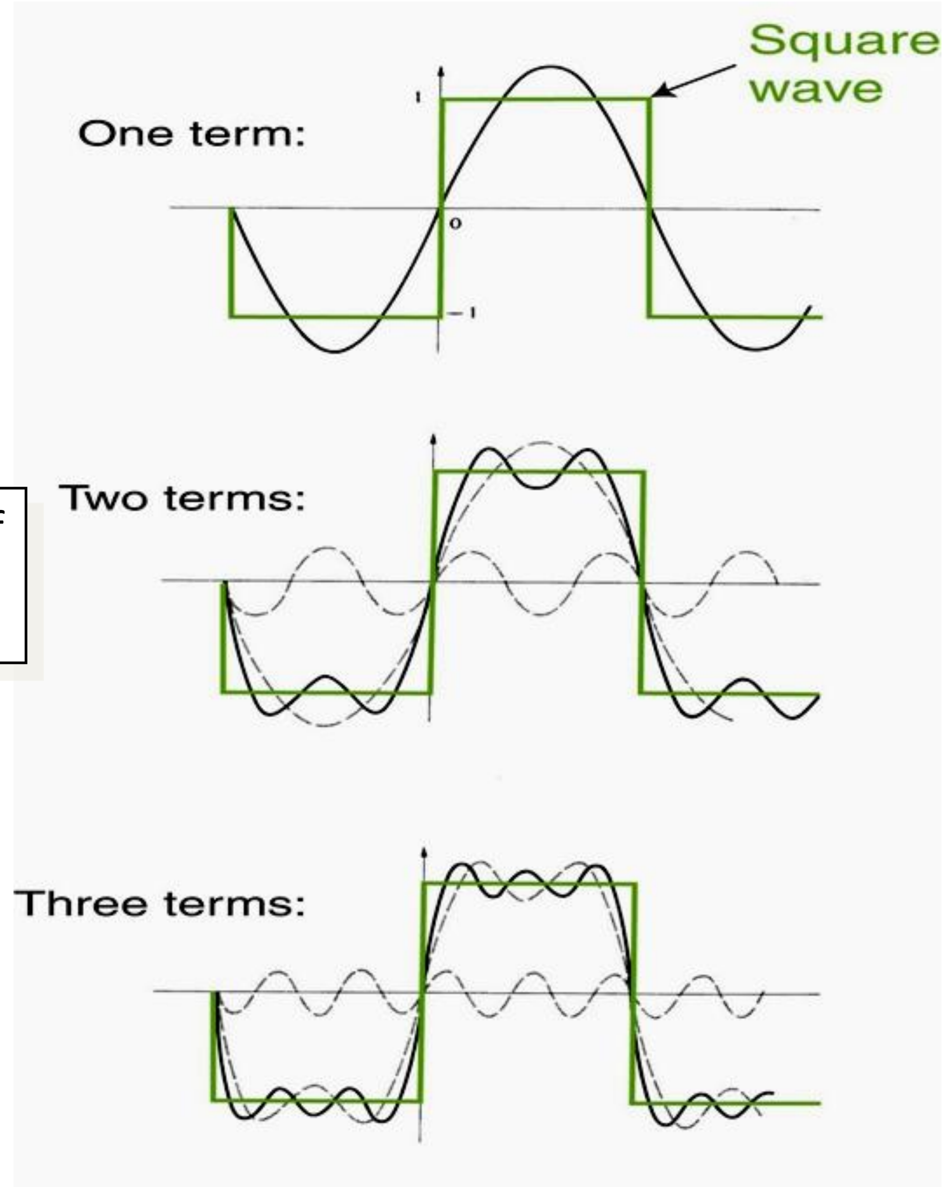
Nobody paid much attention when the work was first published
One of the most important mathematical theories in modern engineering

The big idea …

Any function that periodically repeats itself can be expressed as a sum of sines and cosines of different frequencies each multiplied by a different coefficient – a *Fourier series*

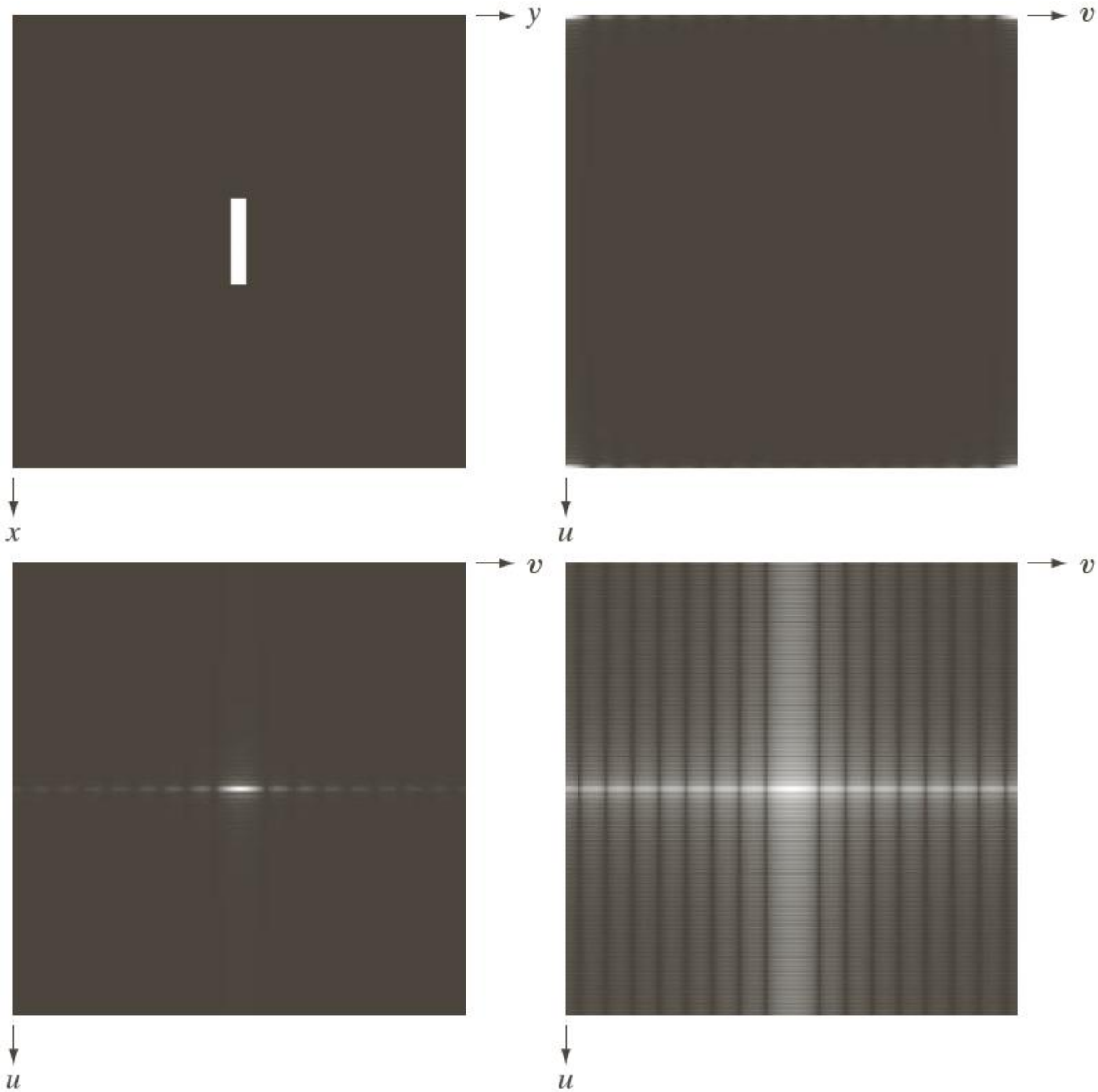


The big idea…



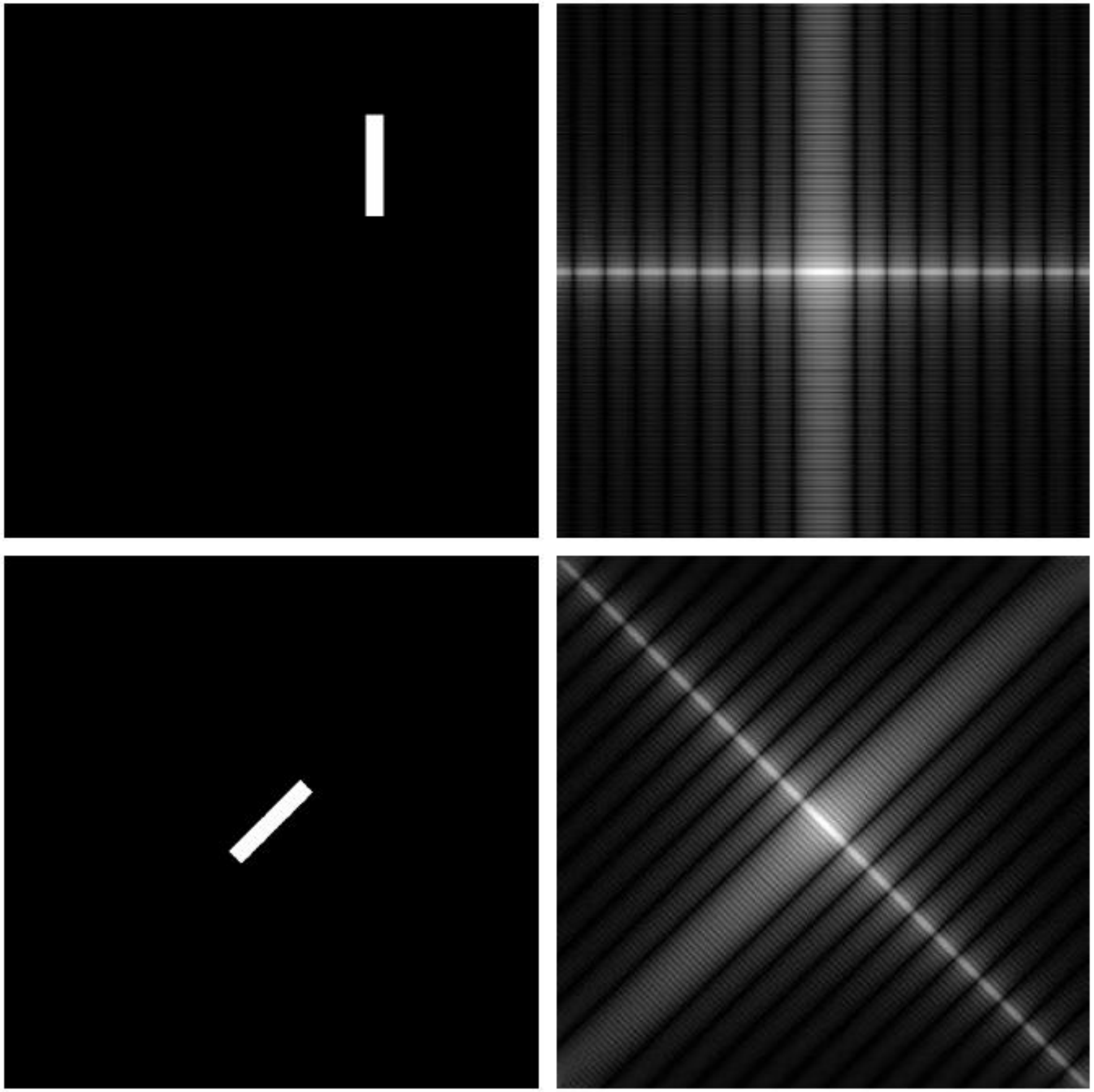
Approximating a square wave as the sum of sine waves

Frequencies in Images



a b
c d

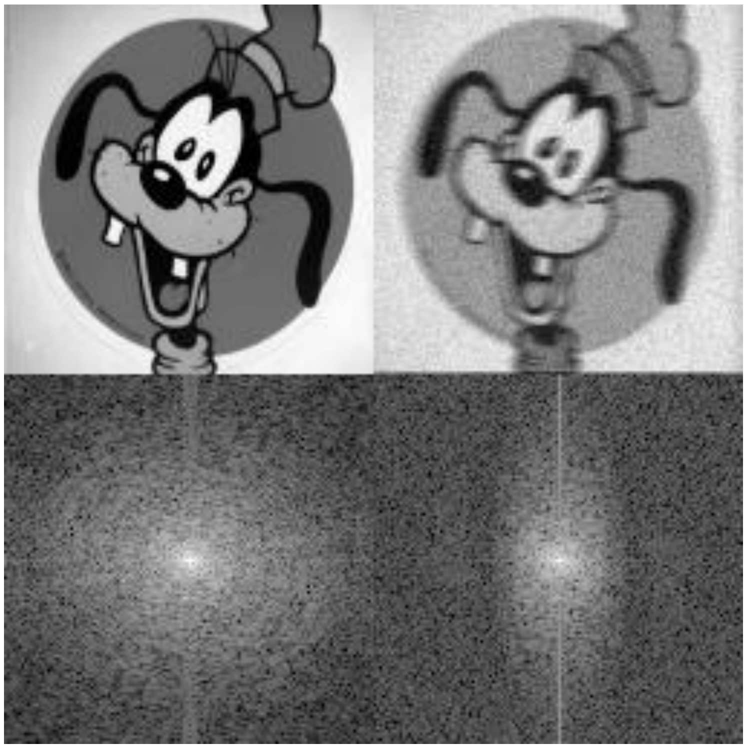
FIGURE 4.24
(a) Image.
(b) Spectrum showing bright spots in the four corners.
(c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.

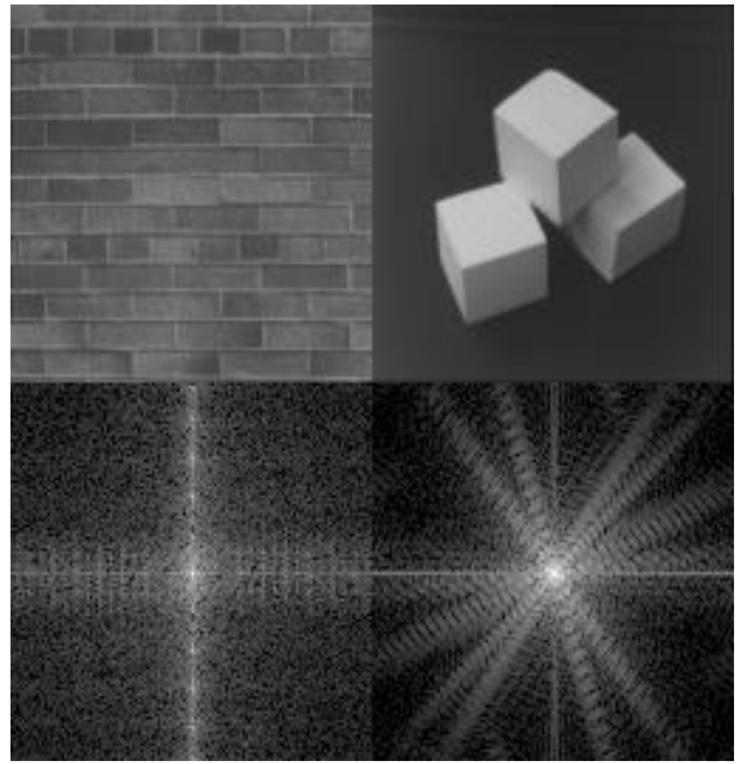


a	b
c	d

FIGURE 4.25
(a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).

Frequencies in Images





Basic 2D FT

$$x(n_1, n_2) = e^{j(\omega_1' n_1 + \omega_2' n_2)} \longrightarrow \boxed{h(n_1, n_2)} \longrightarrow y(n_1, n_2) = ?$$

(ω_1', ω_2')

$$y(n_1, n_2) = x(n_1, n_2) ** h(n_1, n_2)$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} e^{j\omega_1'(n_1-k_1)} e^{j\omega_2'(n_2-k_2)} h(k_1, k_2)$$

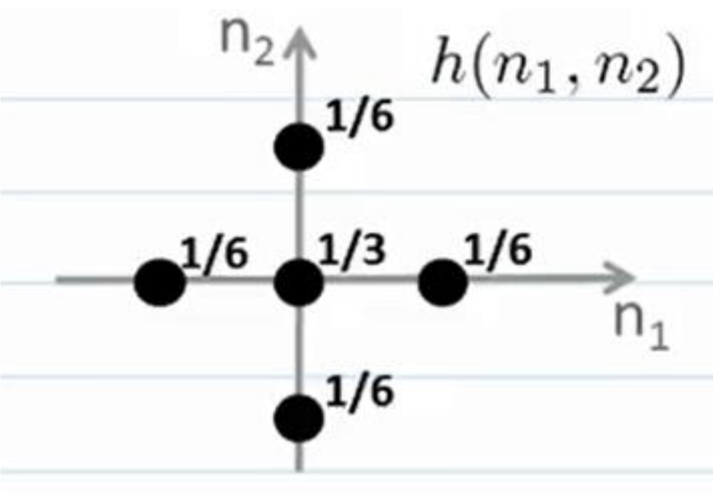
$$= \underbrace{e^{j\omega_1' n_1} e^{j\omega_2' n_2}} \underbrace{\sum_{k_1} \sum_{k_2} h(k_1, k_2) e^{-j\omega_1' k_1} e^{-j\omega_2' k_2}}_{H(\omega_1', \omega_2') \triangleq \text{frequency response}}$$

2D FT

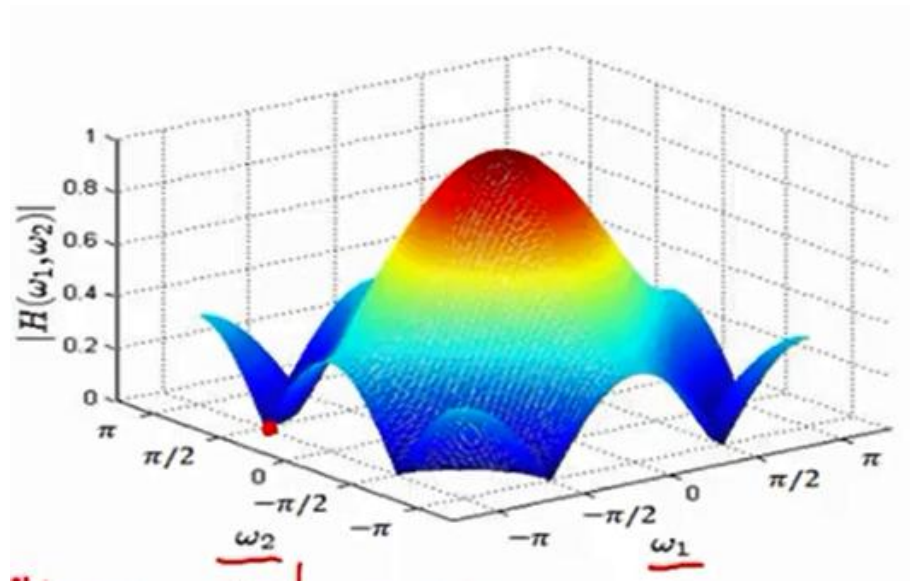
$$X(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

$$x(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) e^{j\omega_1 n_1} e^{j\omega_2 n_2} d\omega_1 d\omega_2$$

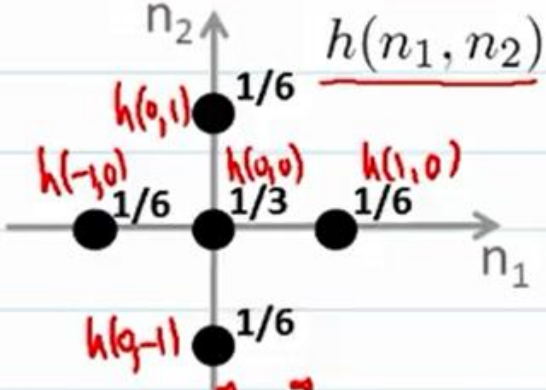
Example

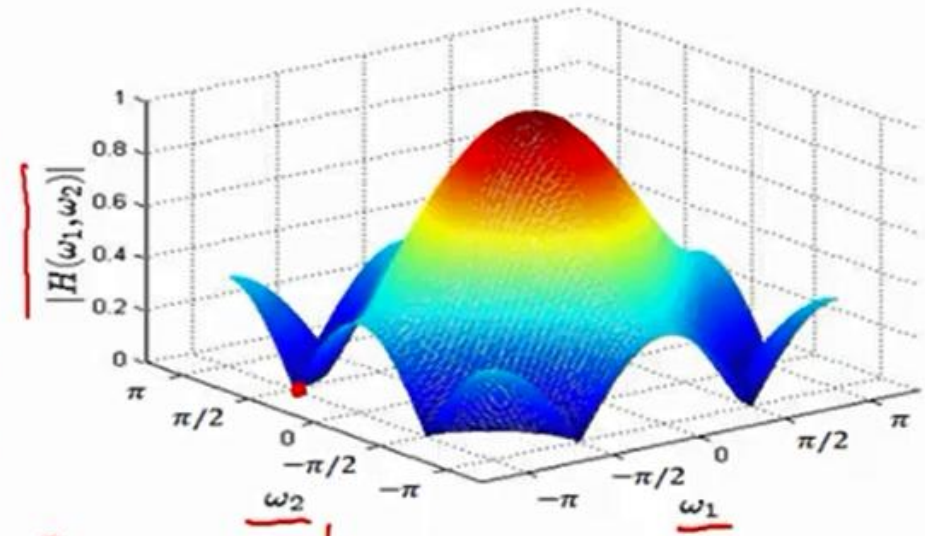


n_2
$h(n_1, n_2)$
1/6
1/6
1/3
1/6
n_1
1/6



$|H(\omega_1,\omega_2)|$
1
0.8
0.6
0.4
0.2
0
π
$\pi/2$
0
$-\pi/2$
$-\pi$
ω_2
$-\pi$
$-\pi/2$
0
$\pi/2$
π
ω_1





$$H(\omega_1,\omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} h(n_1,n_2)\, e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

$$= h(0,0) + h(-1,0)e^{j\omega_1} + h(1,0)e^{-j\omega_1} + h(0,-1)e^{j\omega_2} + h(0,1)e^{-j\omega_2}$$

$$= \frac{1}{3} + \frac{1}{6}e^{j\omega_1} + \frac{1}{6}e^{-j\omega_1} + \frac{1}{6}e^{+j\omega_2} + \frac{1}{6}e^{-j\omega_2}$$

$$= \frac{1}{3} + \frac{1}{6}2\cos\omega_1 + \frac{1}{6}\cdot 2\cos\omega_2$$

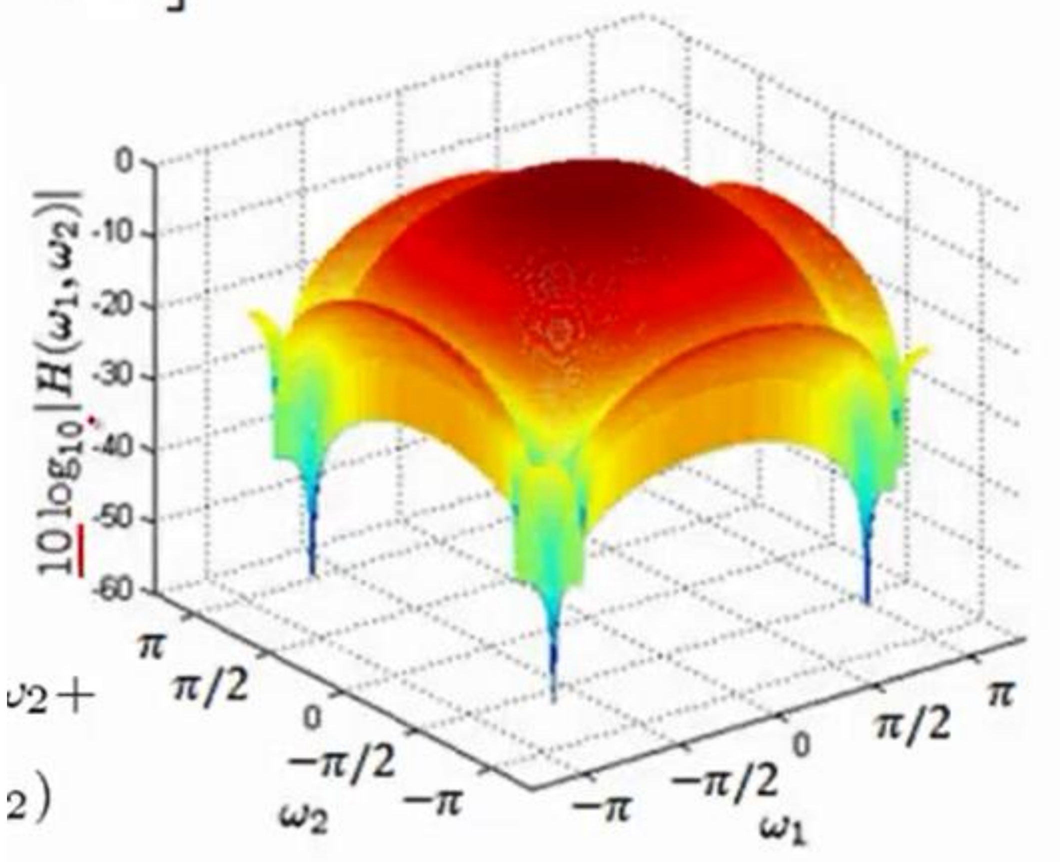
$$= \frac{1}{3}\left(1 + \cos\omega_1 + \cos\omega_2\right) \qquad -1 = 1 \cdot e^{\pm j\pi}$$

$$H(0,0) = 1$$

$$H(-\pi, \pi/2) = 0$$

Example

$$h(n_1, n_2) = \begin{bmatrix} 0.075 & 0.124 & 0.075 \\ 0.124 & 0.204 & 0.124 \\ 0.075 & 0.124 & 0.075 \end{bmatrix}$$



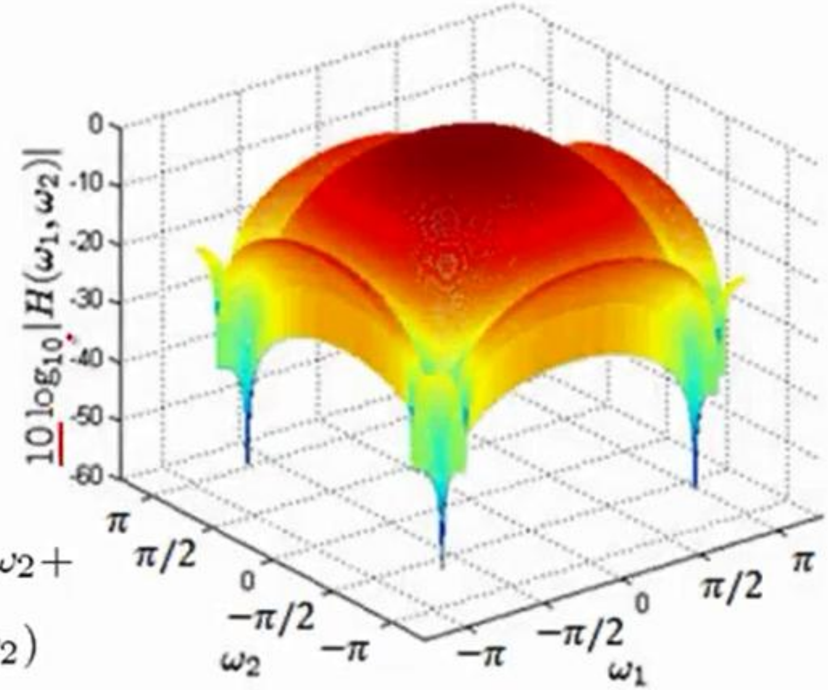
0.124 · 2 · cos w1

h(-1,1) h(1,1) h(0,0)

$$h(n_1, n_2) = \begin{bmatrix} 0.075 & 0.124 & 0.075 \\ 0.124 & 0.204 & 0.124 \\ 0.075 & 0.124 & 0.075 \end{bmatrix}$$

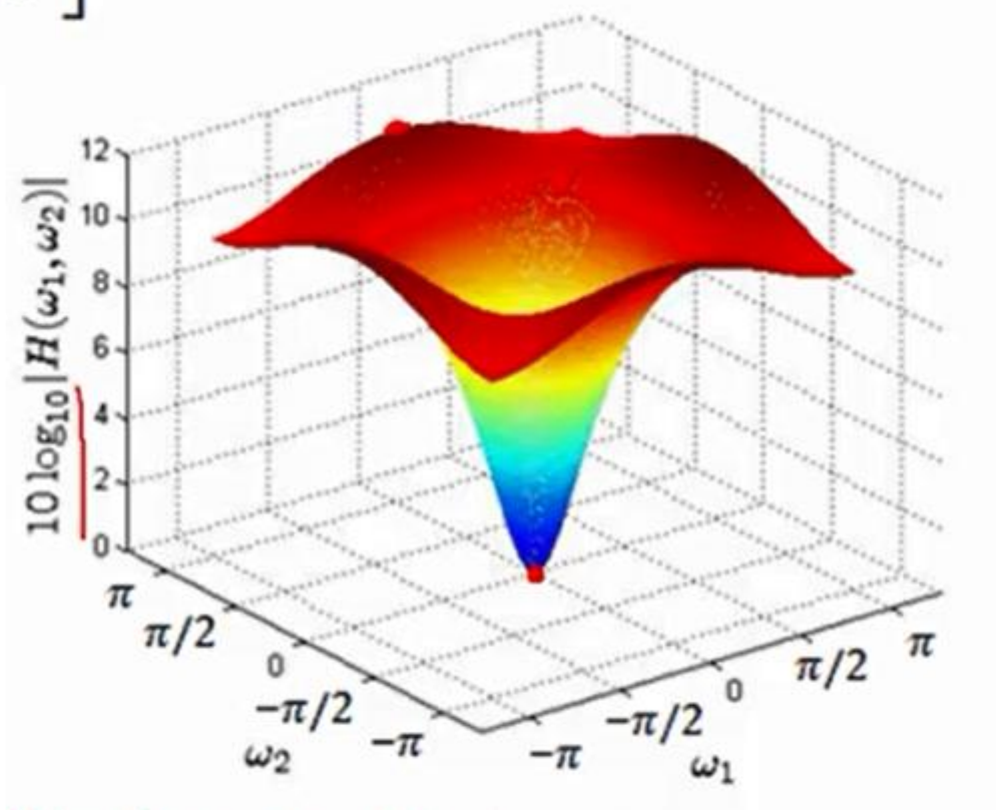
0.075 · 2 · cos(w1+w2)

$$H(\omega_1, \omega_2) = 0.204 + 0.124 \cdot 2 \cdot cos\omega_1 + 0.124 \cdot 2 \cdot cos\omega_2 + \\ 0.075 \cdot 2 \cdot cos(\omega_1 + \omega_2) + 0.075 \cdot 2 \cdot cos(\omega_1 - \omega_2)$$



Example

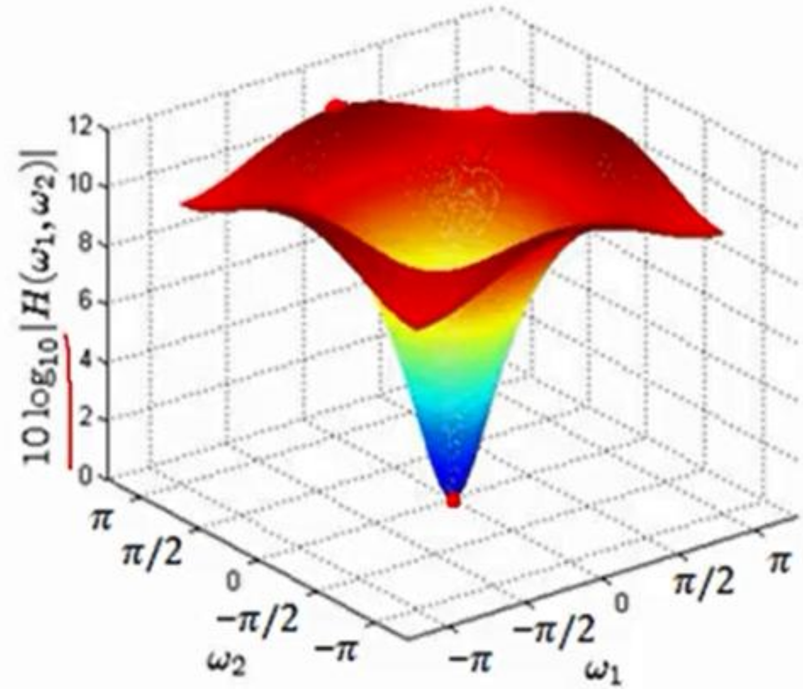
$$h(n_1, n_2) = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 9 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$



$$h(n_1, n_2) = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 9 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

h(0,0)

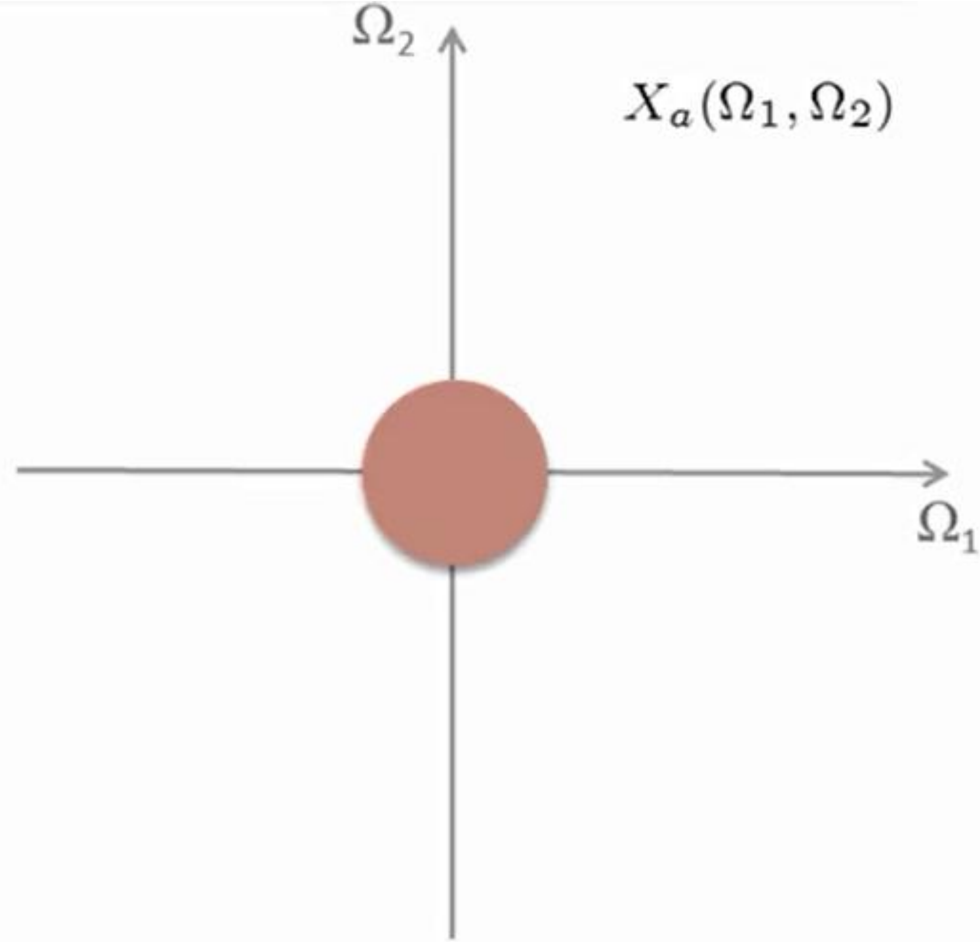
$$H(\omega_1, \omega_2) = 9 - 2 \cdot cos\omega_1 - 2 \cdot cos\omega_2 - 2 \cdot cos(\omega_1 + \omega_2) - 2 \cdot cos(\omega_1 - \omega_2)$$



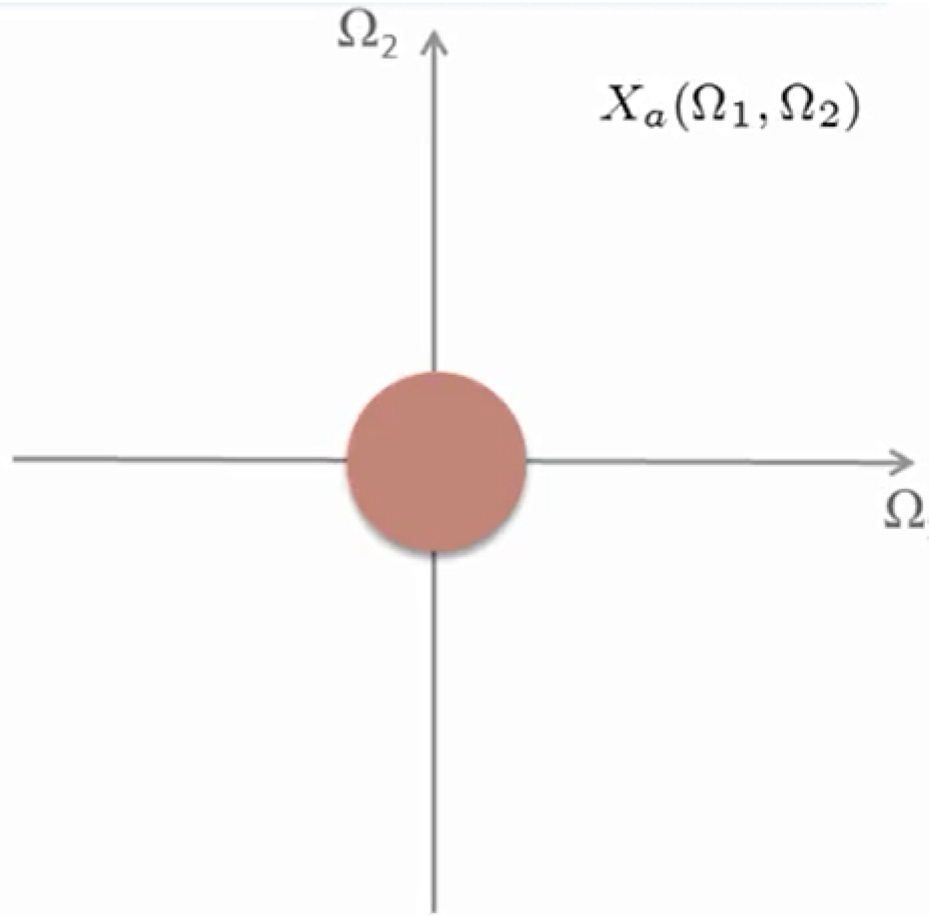
$H(0,0)=1 \rightarrow \log H(0,0) = 0$

$H(0,\pi) = 13, \; H(\pi,\pi) = 9$

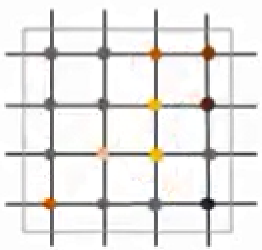
2D Sampling

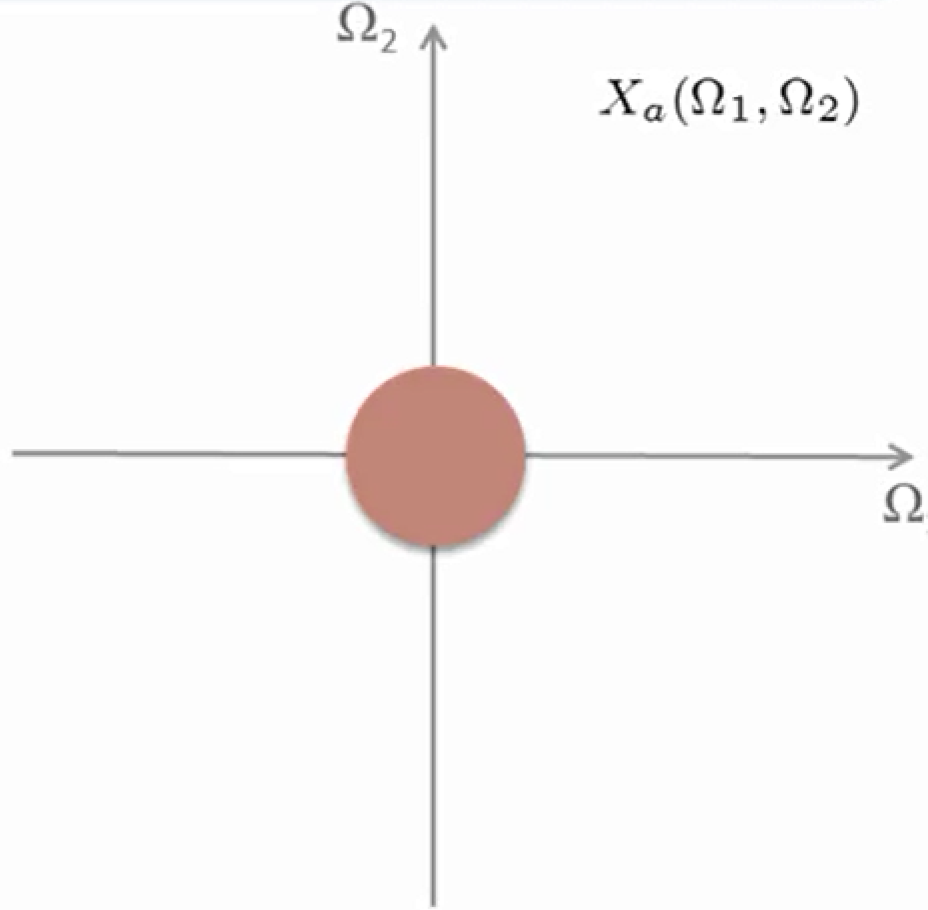


2D Sampling

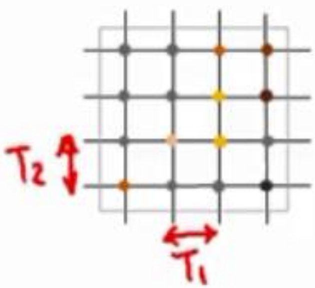


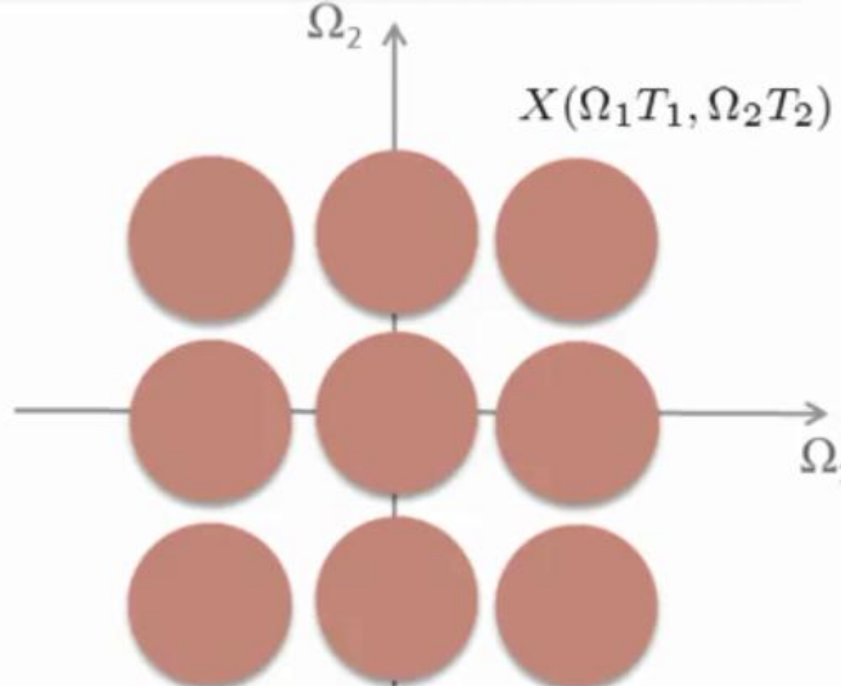
2D Sampling





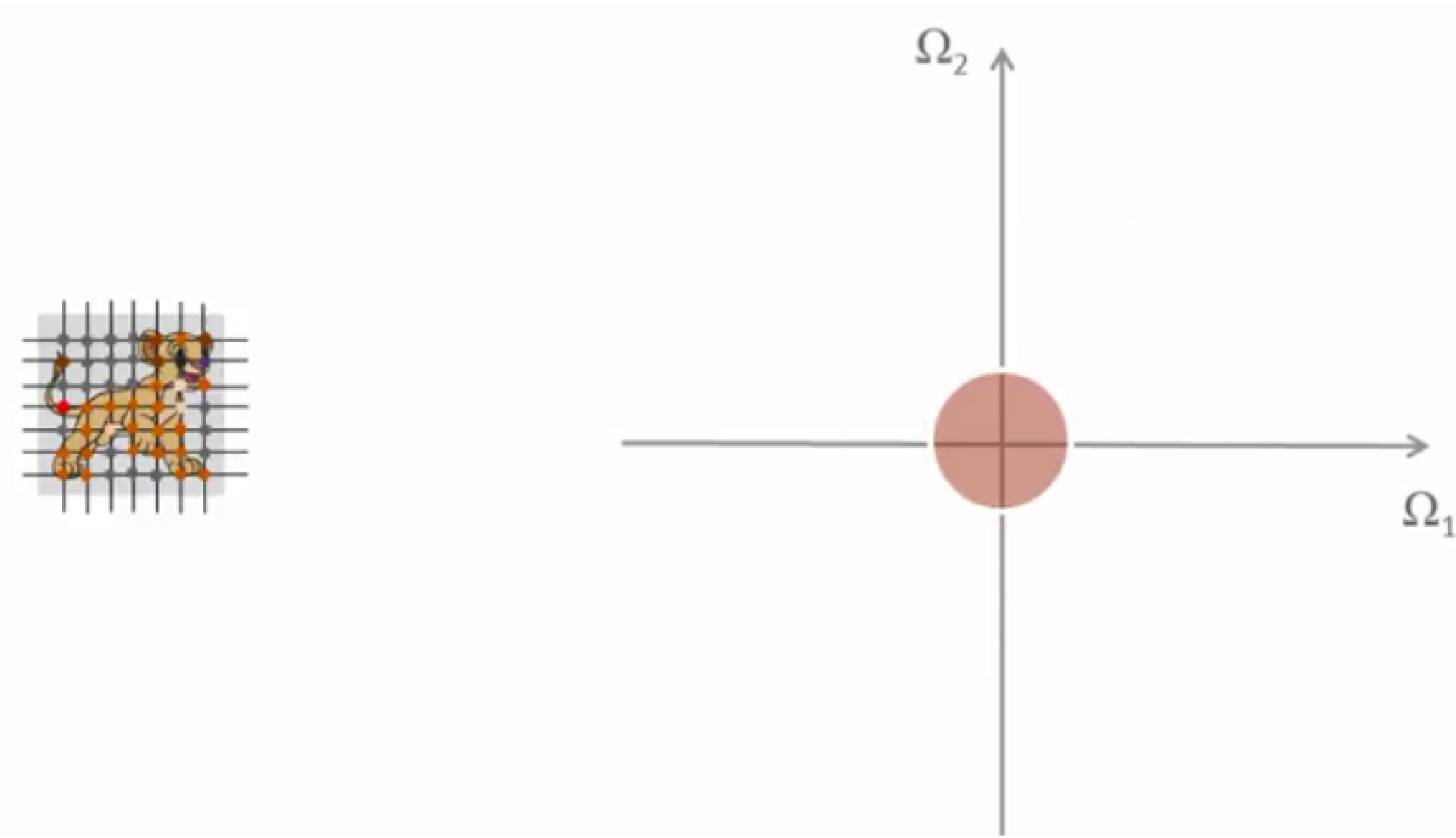
2D Sampling



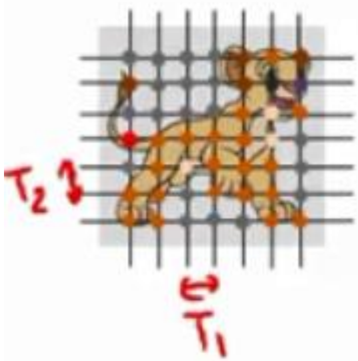


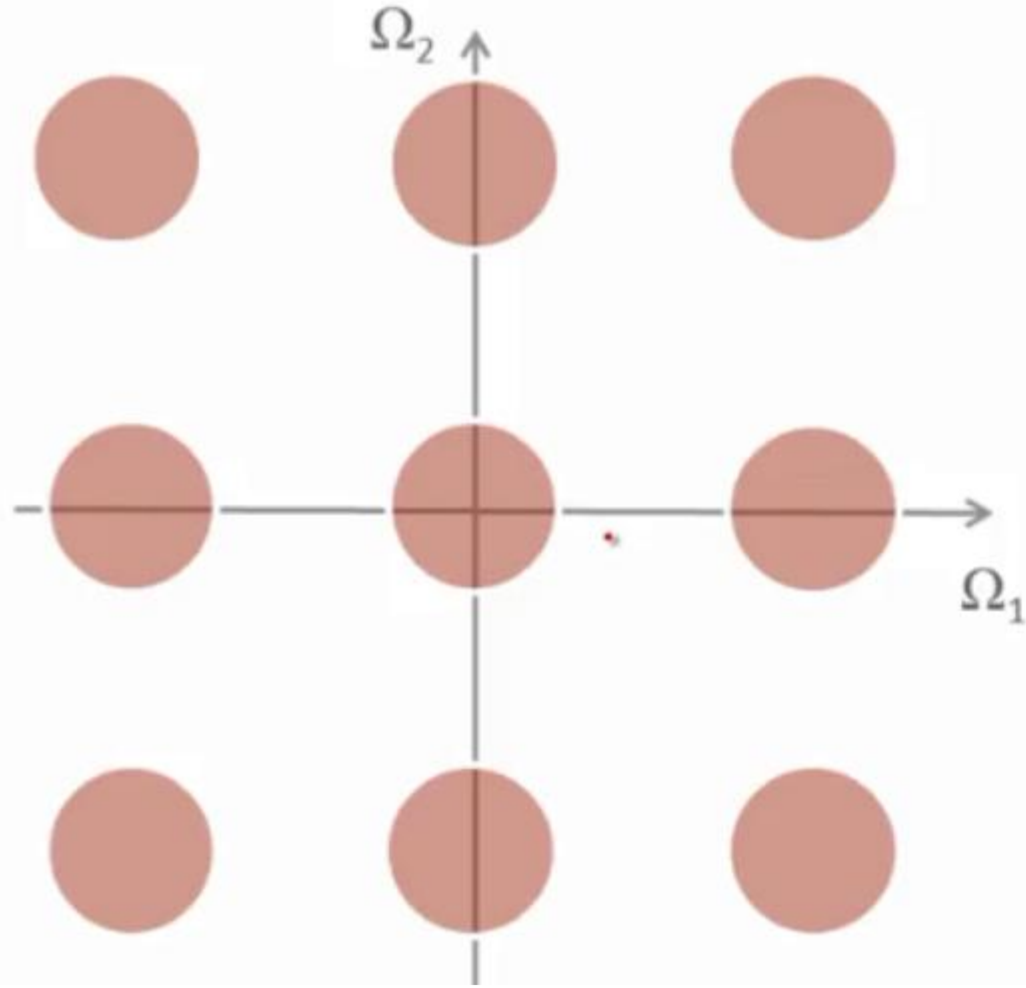
$$X(\Omega_1 T_1, \Omega_2 T_2) = \frac{1}{T_1 T_2} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} X_a(\Omega_1 - \frac{2\pi}{T_1} k_1, \Omega_2 - \frac{2\pi}{T_2} k_2)$$

Over Sampling

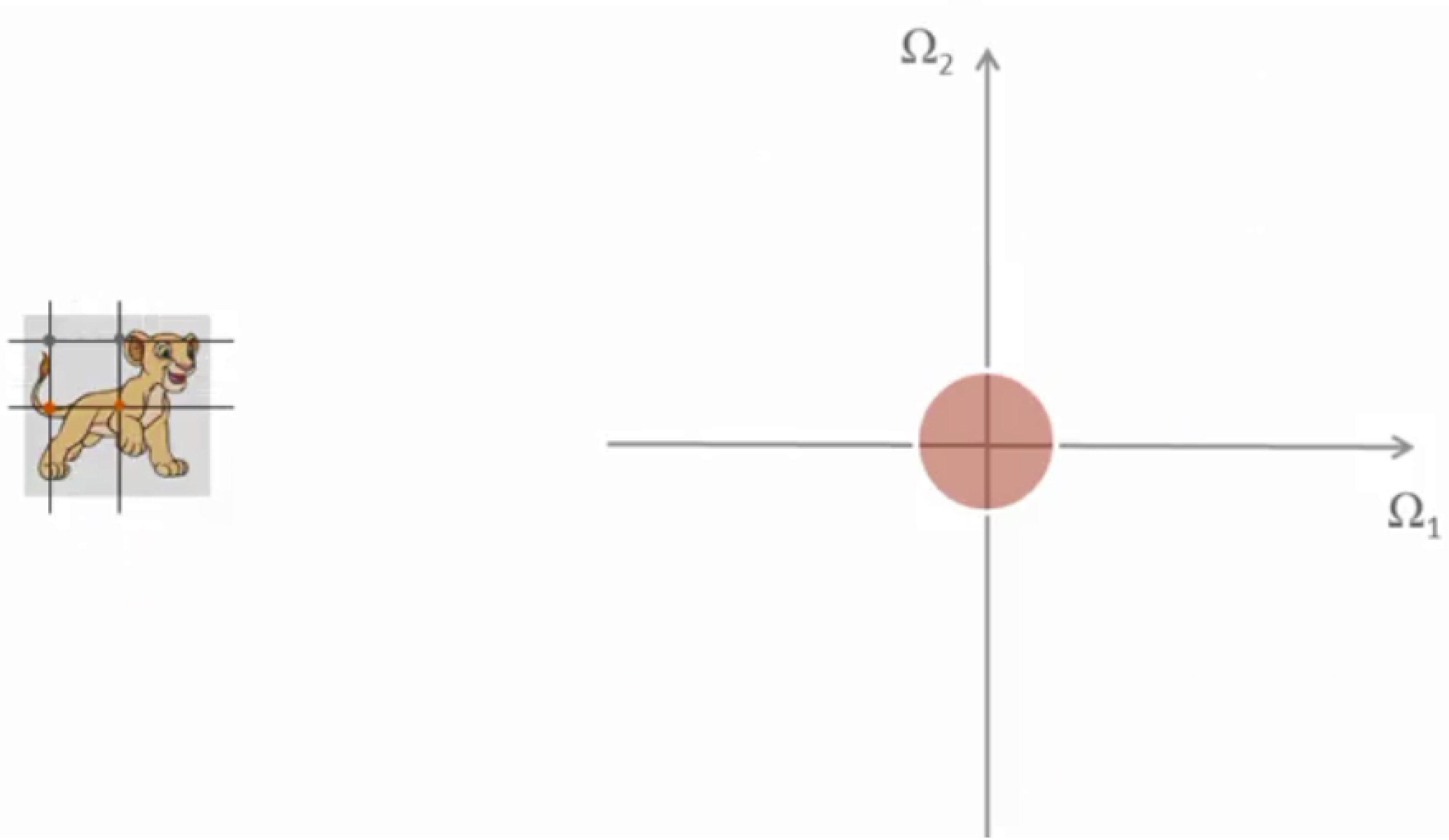


Over Sampling

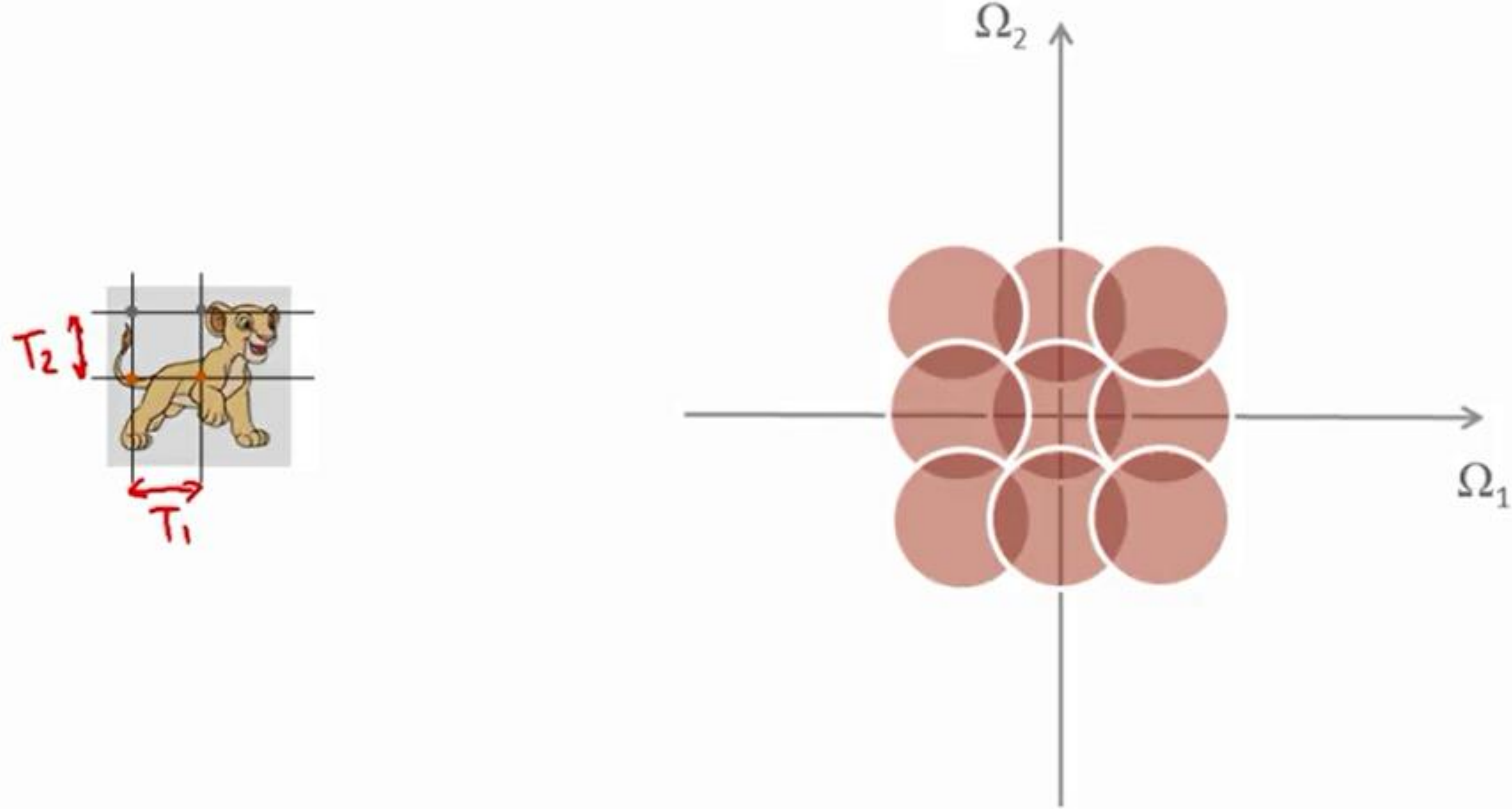




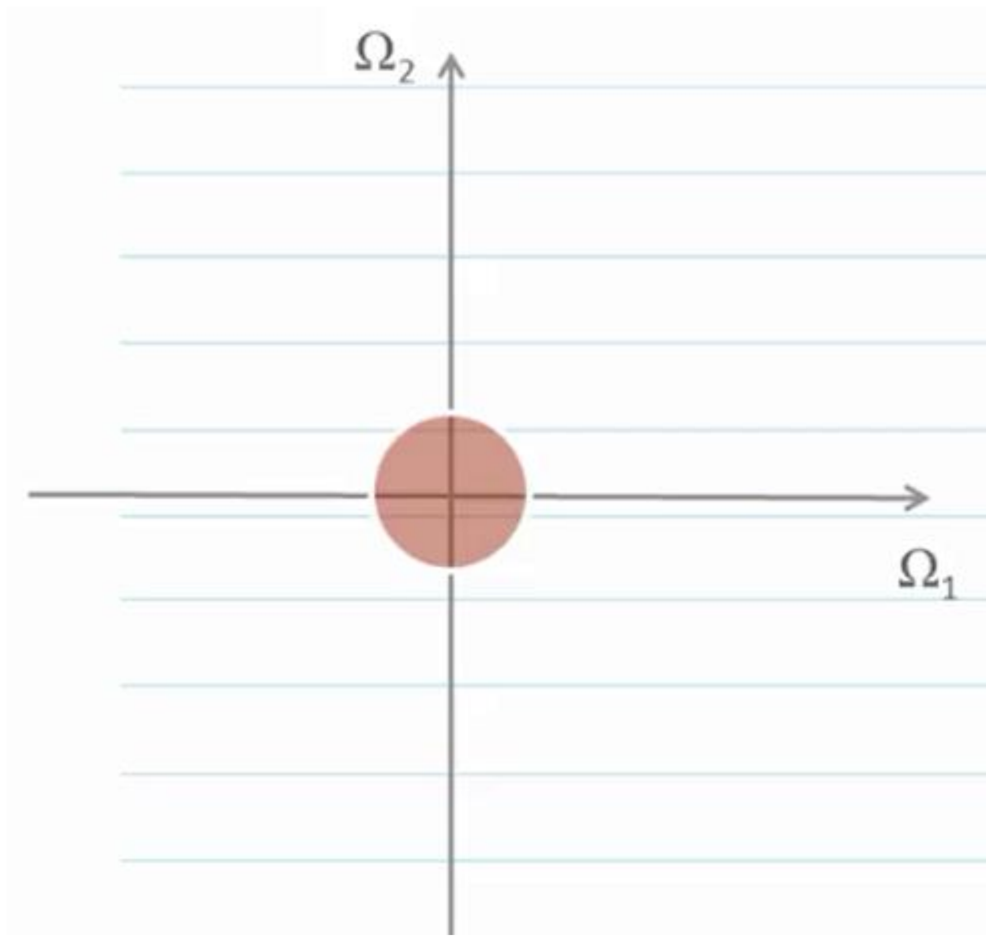
Under Sampling



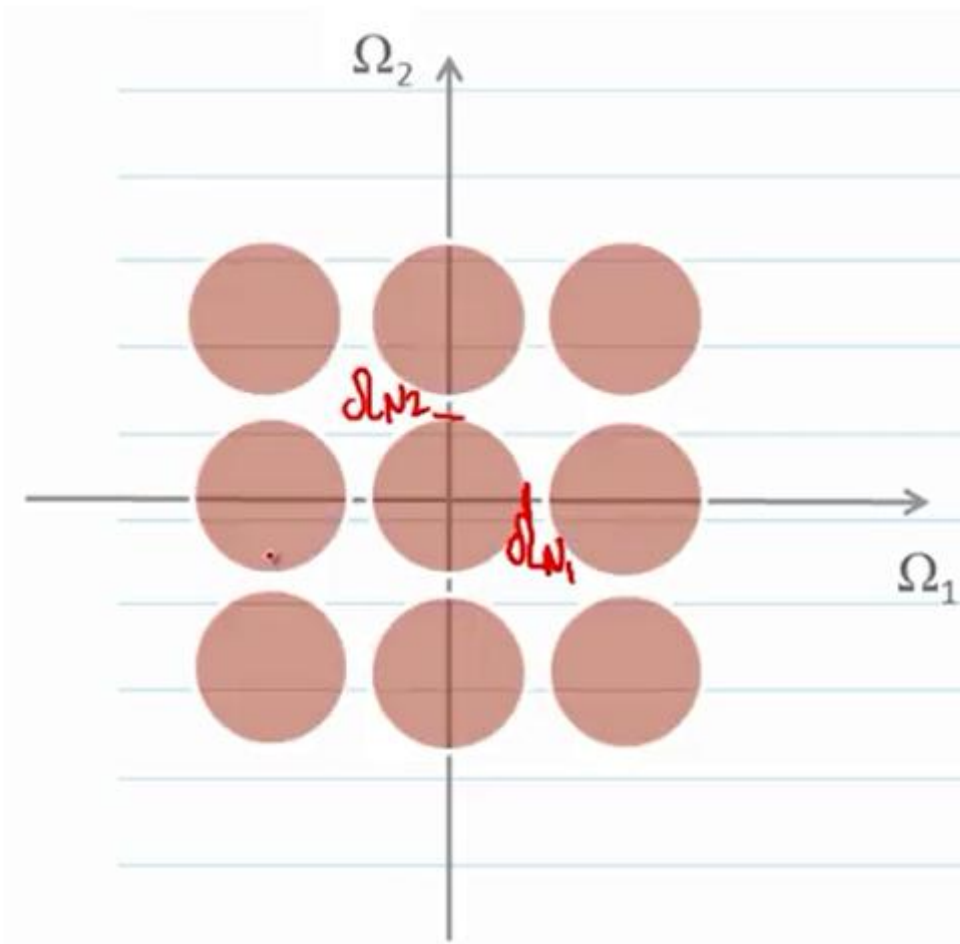
Under Sampling



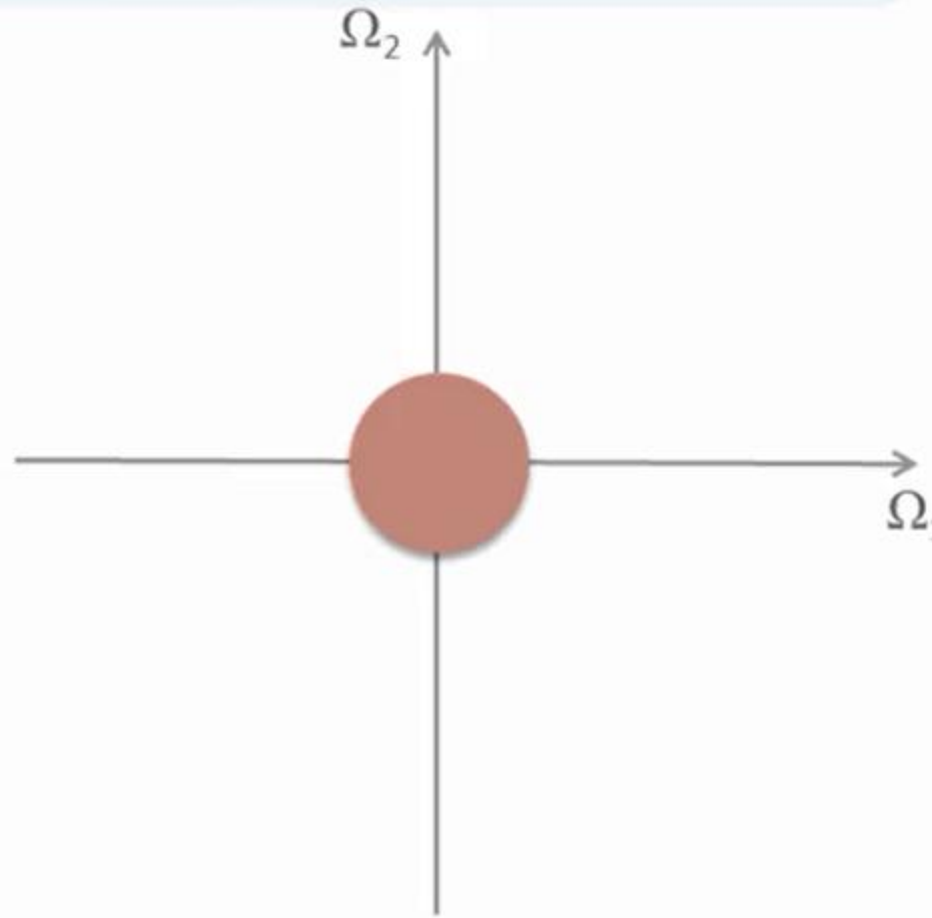
2D Nyquist Theorem



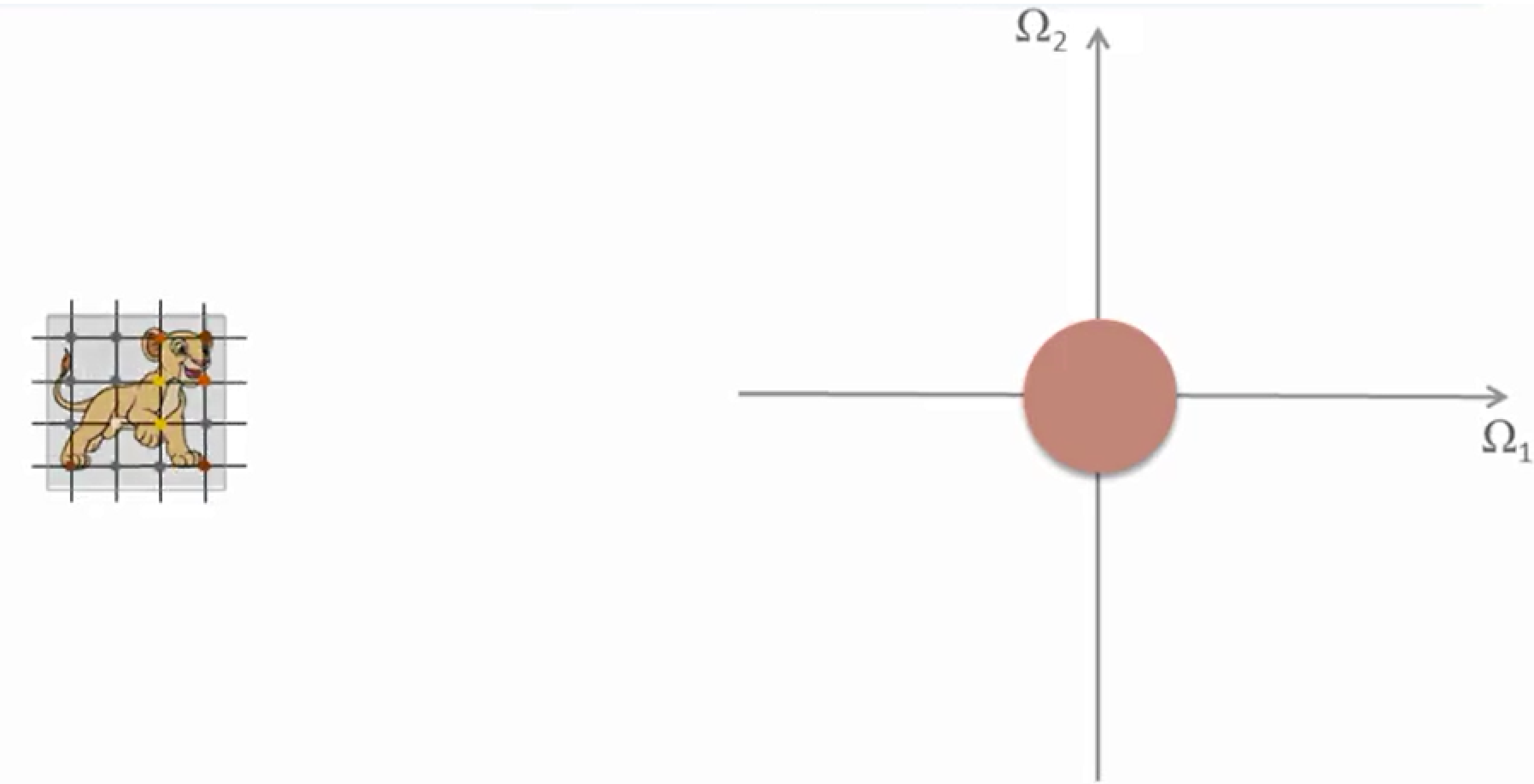
2D Nyquist Theorem



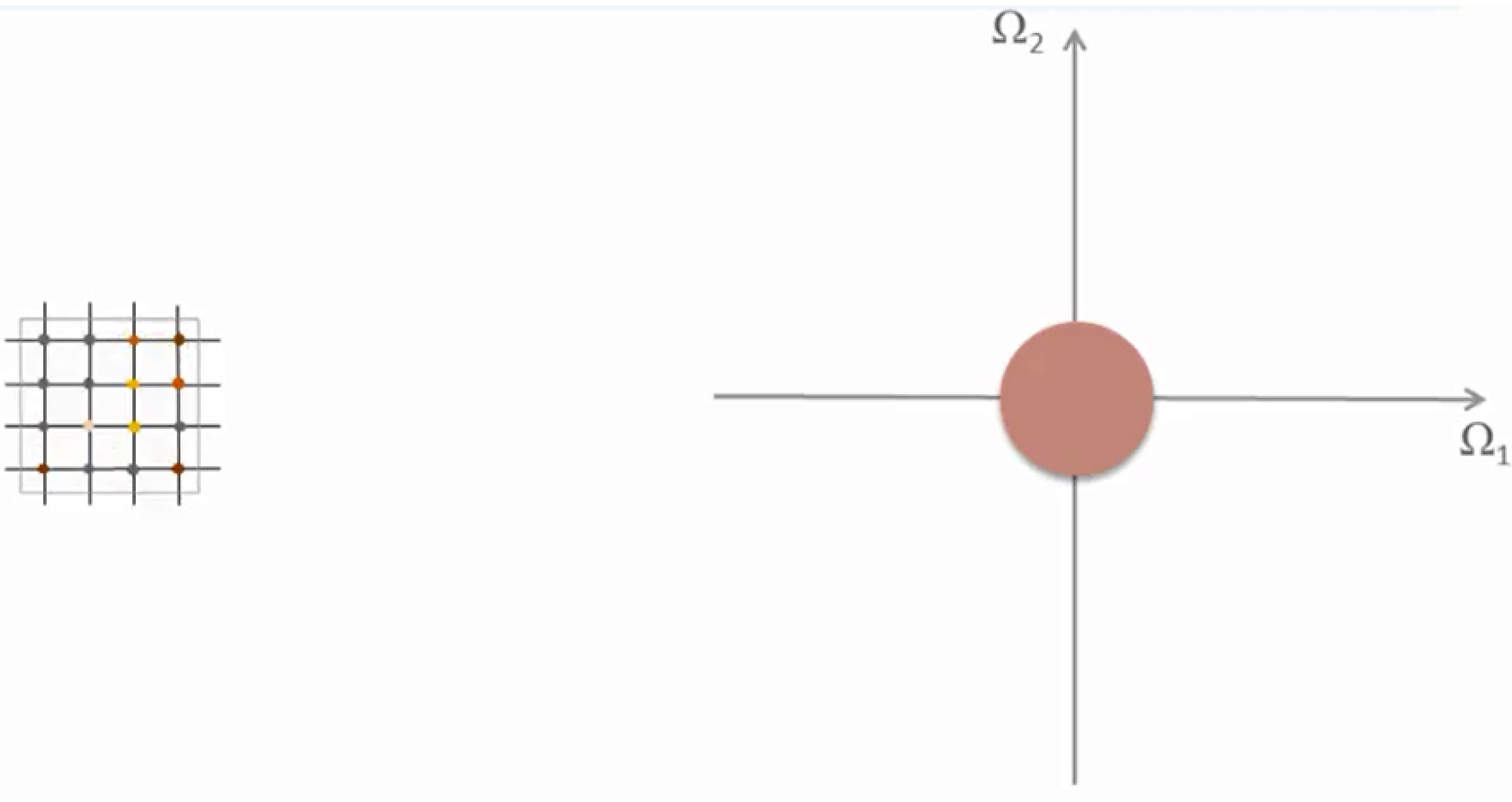
Sampling in the Frequency



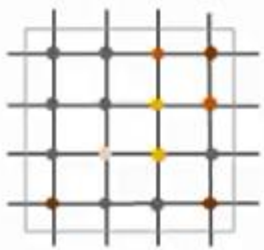
Sampling in the Frequency

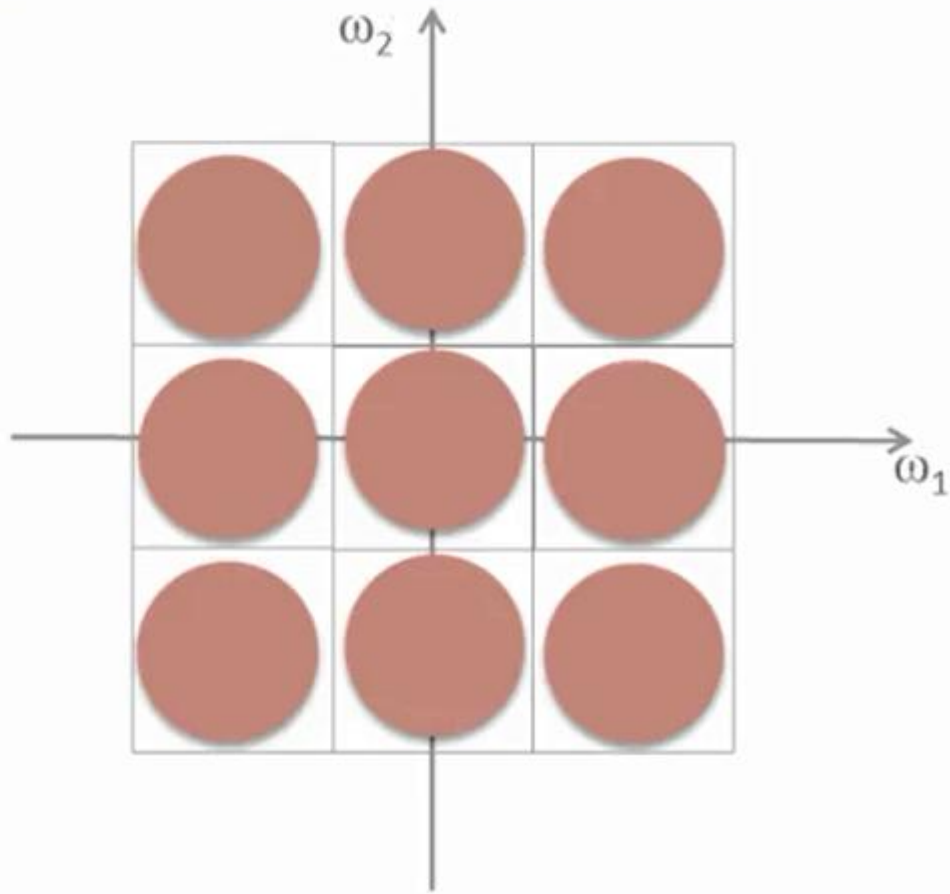


Sampling in the Frequency

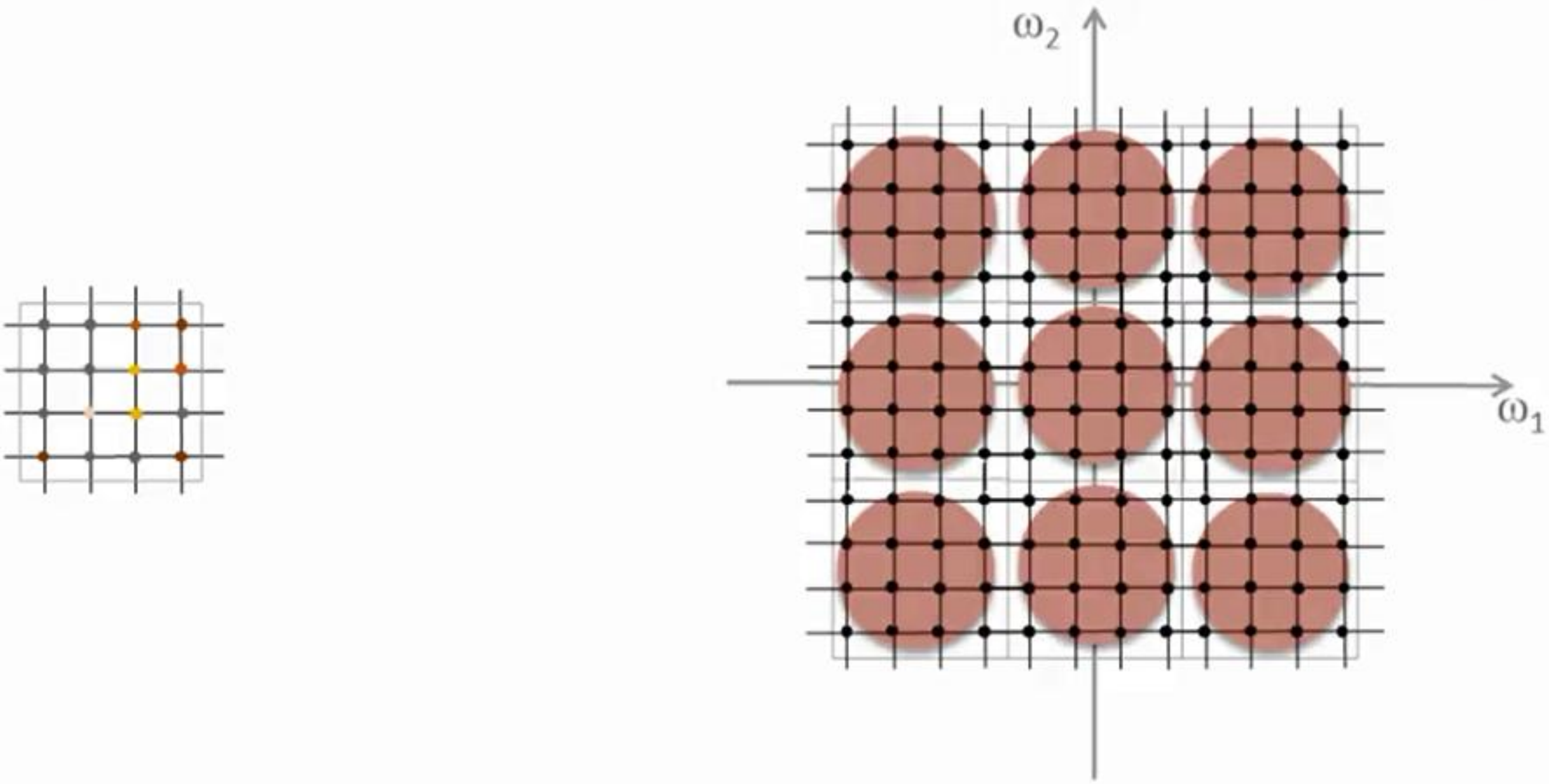


Sampling in the Frequency

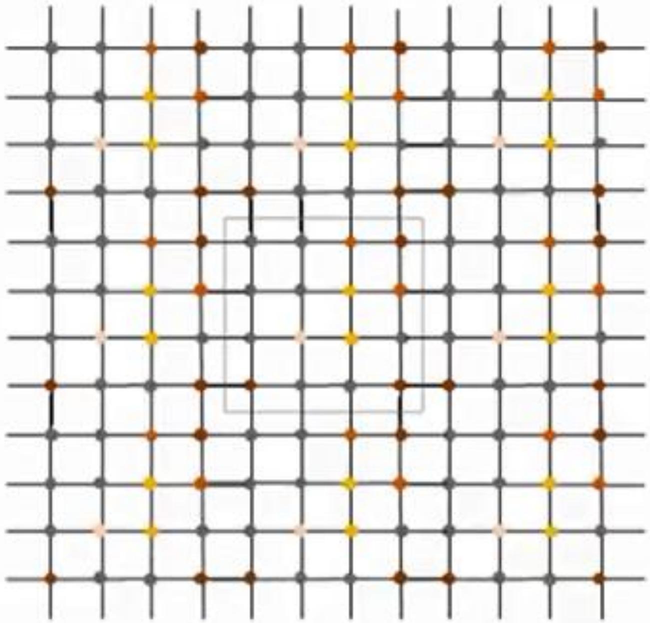


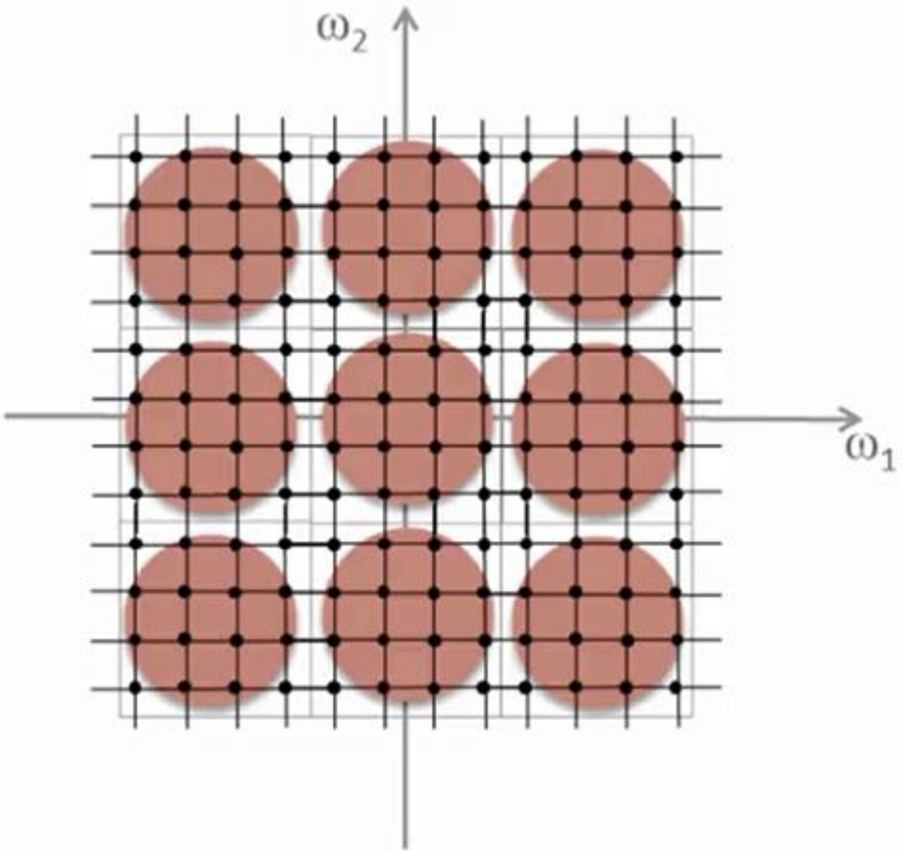


Sampling in the Frequency

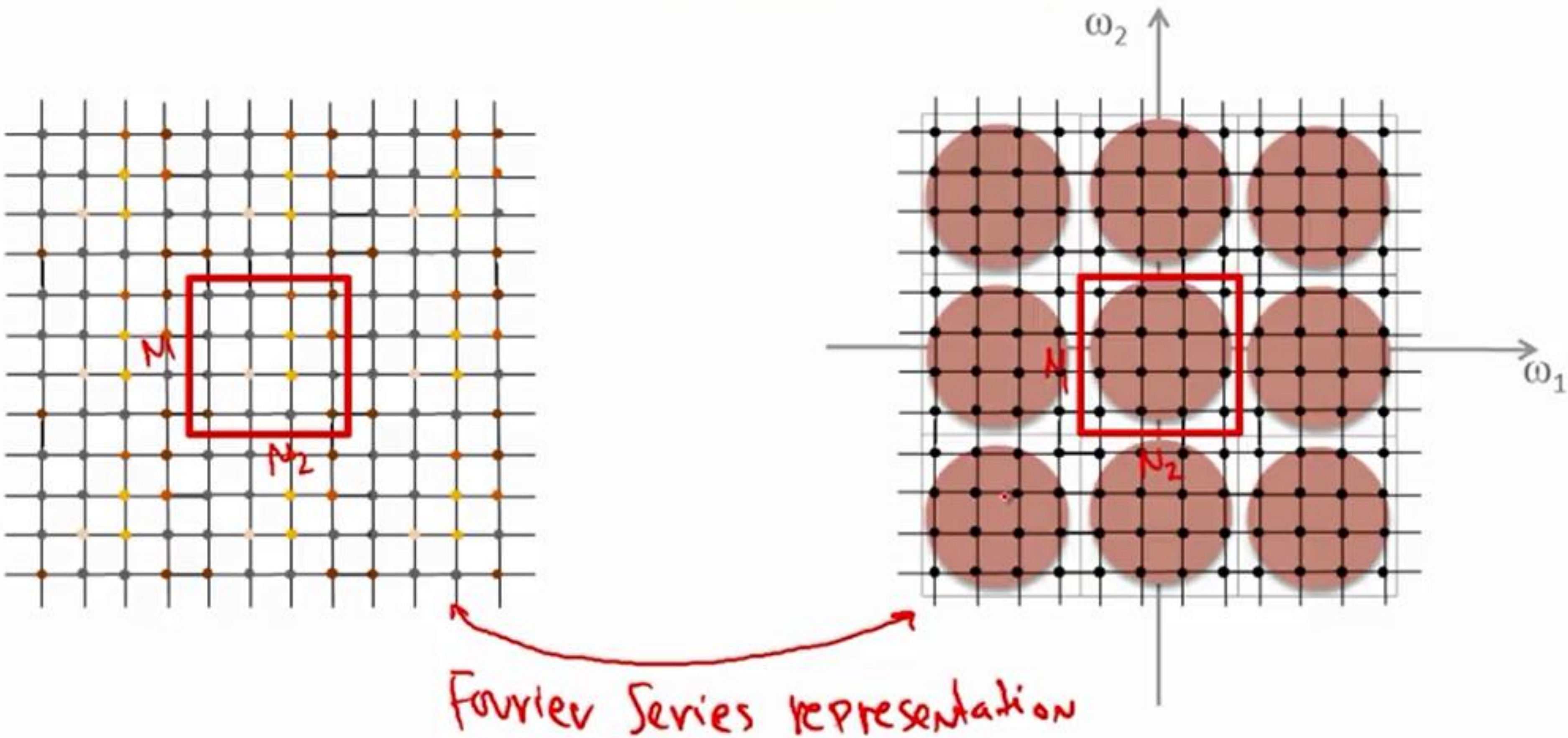


Sampling in the Frequency





Sampling in the Frequency



2DFT to 2D DFT

$$X(\omega_1, \omega_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

$$X(k_1, k_2) = X(\omega_1, \omega_2)|_{\omega_1 = \frac{2\pi}{N_1} k_1, \omega_2 = \frac{2\pi}{N_2} k_2} \quad \begin{cases} k_1 = 0, \ldots, N_1 - 1 \\ k_2 = 0, \ldots, N_2 - 1 \end{cases}$$

$$X(k_1, k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) e^{-j\frac{2\pi}{N_1} n_1 k_1} e^{-j\frac{2\pi}{N_2} n_2 k_2} \quad \begin{cases} k_1 = 0, \ldots, N_1 - 1 \\ k_2 = 0, \ldots, N_2 - 1 \end{cases}$$

$$x(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X(k_1, k_2) e^{j\frac{2\pi}{N_1} n_1 k_1} e^{j\frac{2\pi}{N_2} n_2 k_2} \quad \begin{cases} n_1 = 0, \ldots, N_1 - 1 \\ n_2 = 0, \ldots, N_2 - 1 \end{cases}$$

Readings from Book (3rd Edn.)

- Basic of frequency domain

Acknowledgements

- Digital Image Processing", Rafael C. Gonzalez & Richard E. Woods, Addison-Wesley, 2002
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Material in these slides has been taken from, the following resources