Lecture 16
Wavelets

INTRODUCTION TO WAVELETS

WHAT IS A TRANSFORM AND WHY DO WE NEED ONE ?

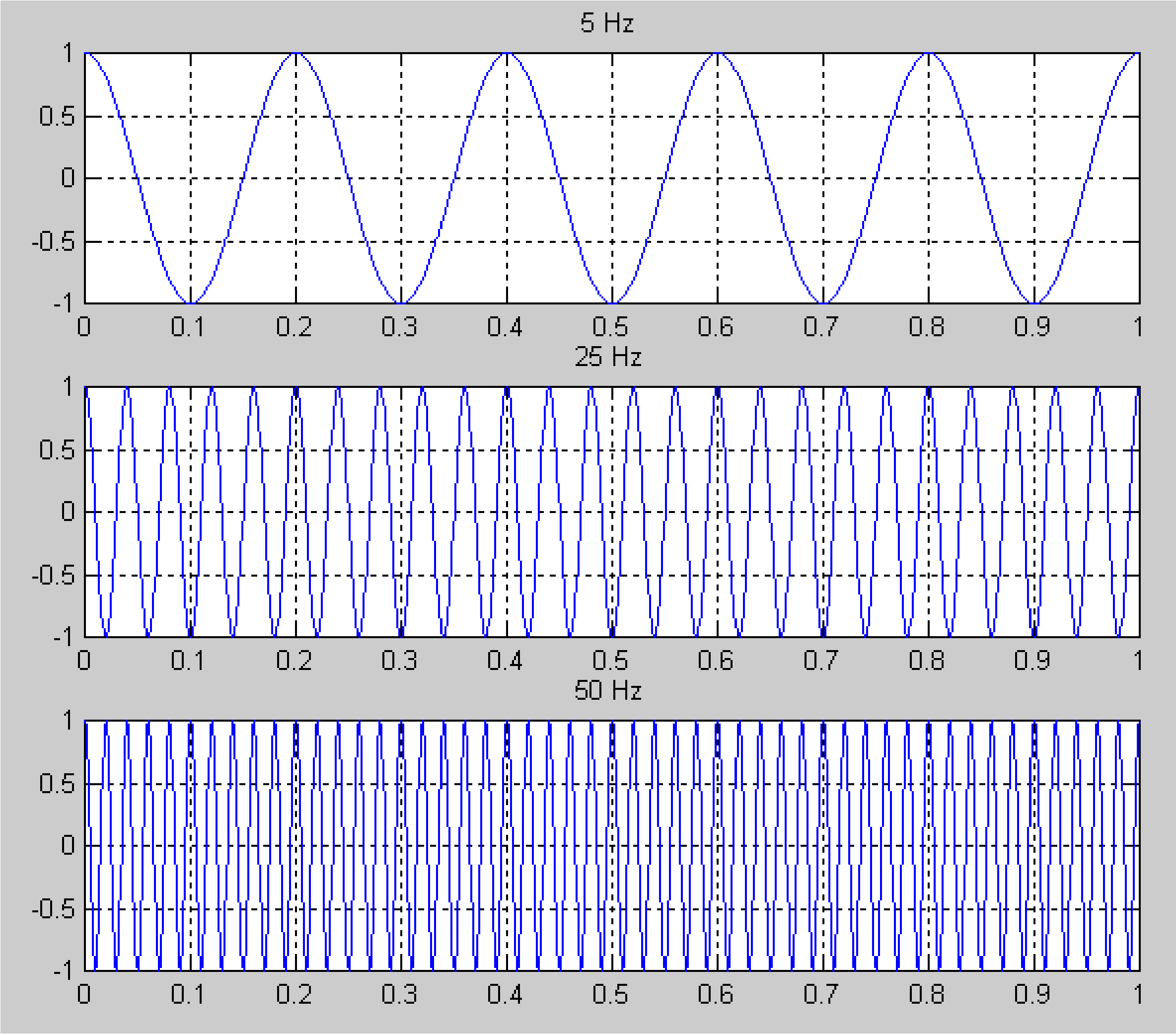
- Transform: **A mathematical operation that takes a function or sequence and maps it into another one**
- **Transforms are good things because…**
  - **The transform of a function may give additional /hidden information about the original function, which may not be available /obvious otherwise**
  - **The transform of an equation may be easier to solve than the original equation**
  - **The transform of a function/sequence may require less storage, hence provide data compression / reduction**
  - **An operation may be easier to apply on the transformed function, rather than the original function**

FT AT WORK

$$x_1(t) = \cos(2\pi \cdot 5 \cdot t)$$

$$x_2(t) = \cos(2\pi \cdot 25 \cdot t)$$

$$x_3(t) = \cos(2\pi \cdot 50 \cdot t)$$

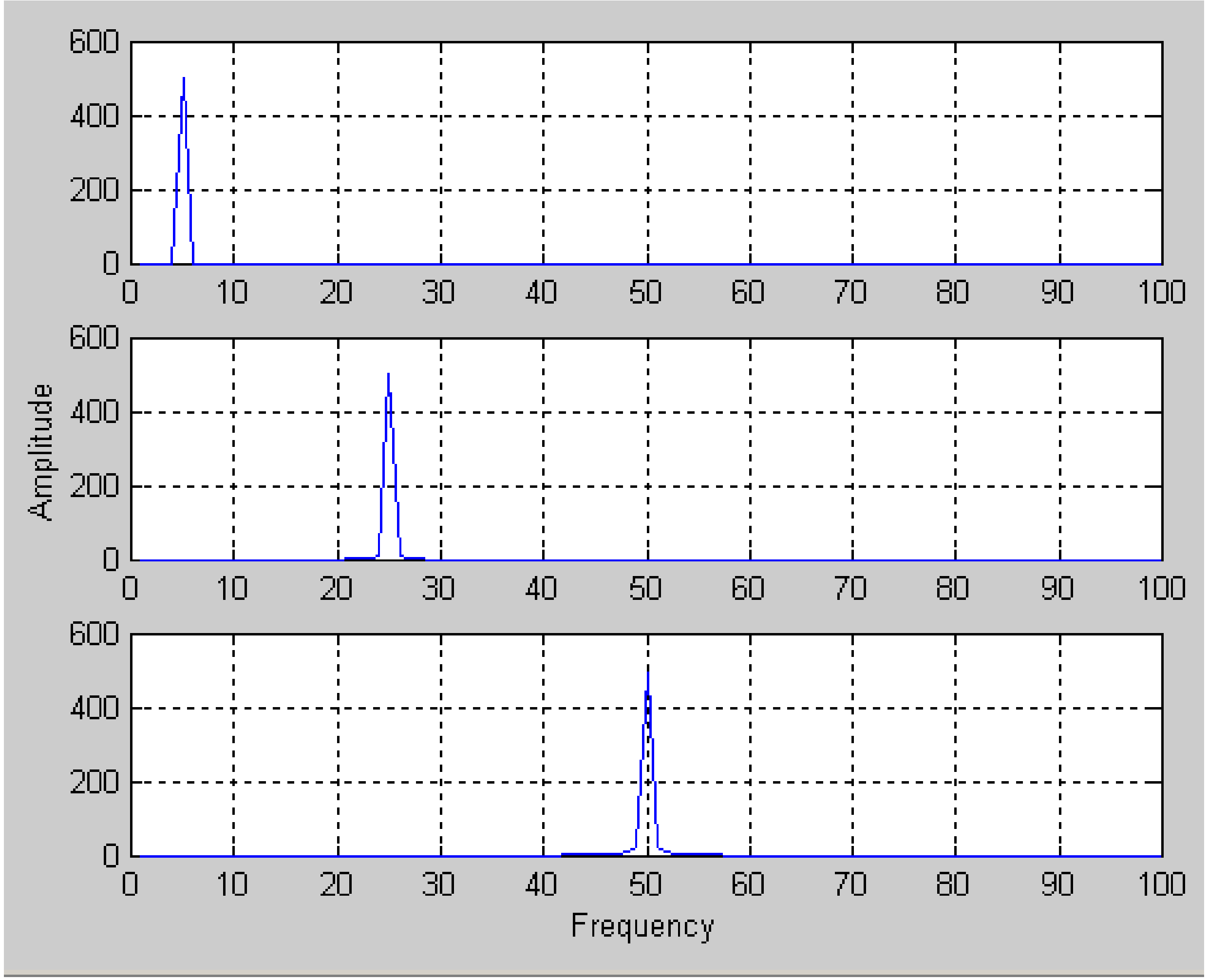


FT AT WORK

$$x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)$$

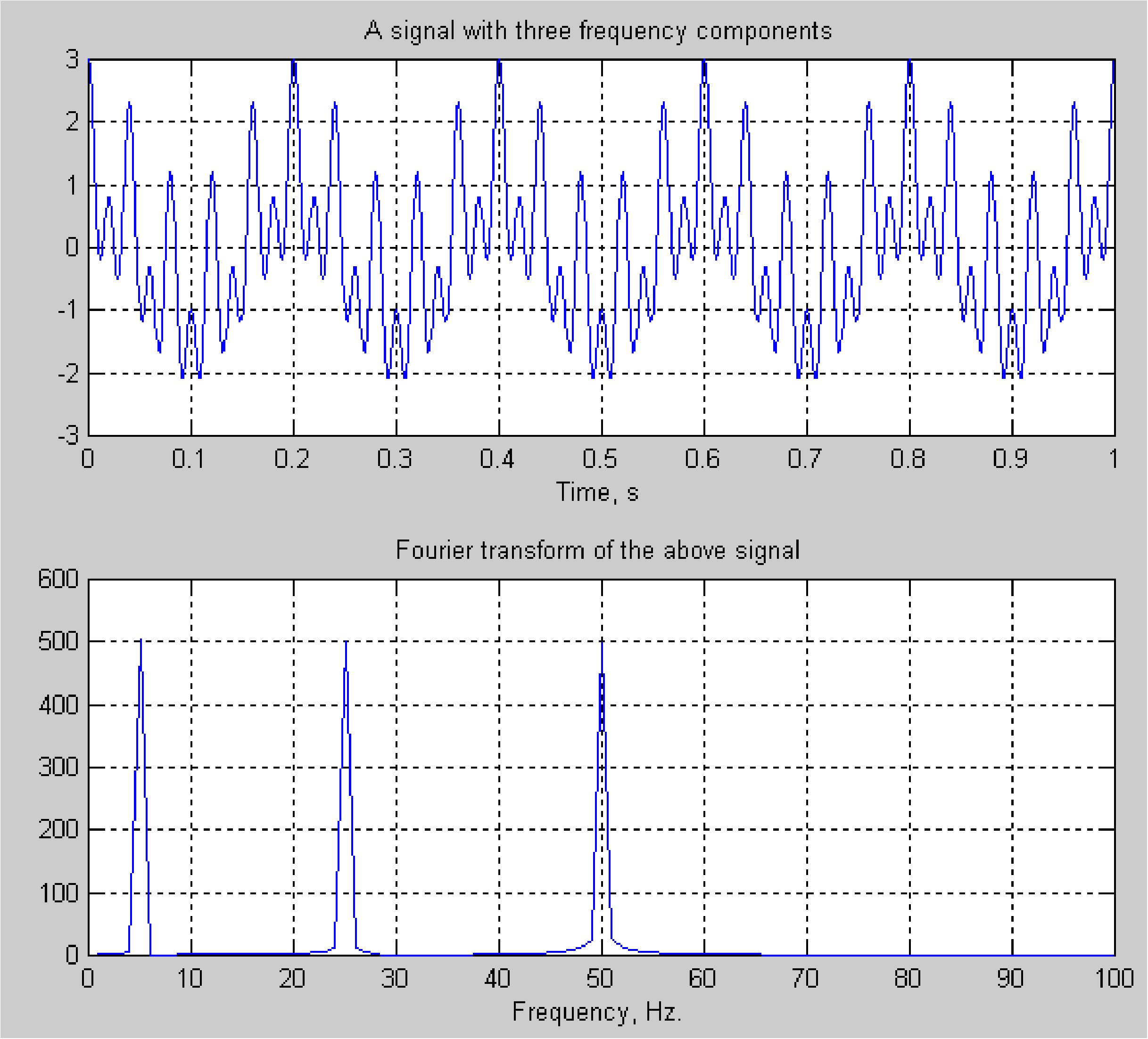
$$x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$$

$$x_3(t) \overset{\mathcal{F}}{\longleftrightarrow} X_3(\omega)$$



FT AT WORK

$$x_4(t) = \cos(2\pi \cdot 5 \cdot t) + \cos(2\pi \cdot 25 \cdot t) + \cos(2\pi \cdot 50 \cdot t)$$



$$x_4(t) \xleftrightarrow{\mathcal{F}} X_4(\omega)$$

Criticism of Fourier Spectrum

It's giving you the spectrum of the 'whole time-series'

Which is OK if the time-series is stationary

But what if its not?

We need a technique that can "march along" a time series and that is capable of:

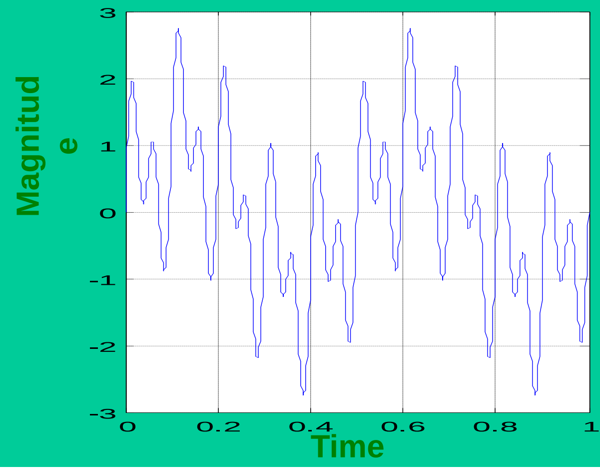
Analyzing spectral content in different places

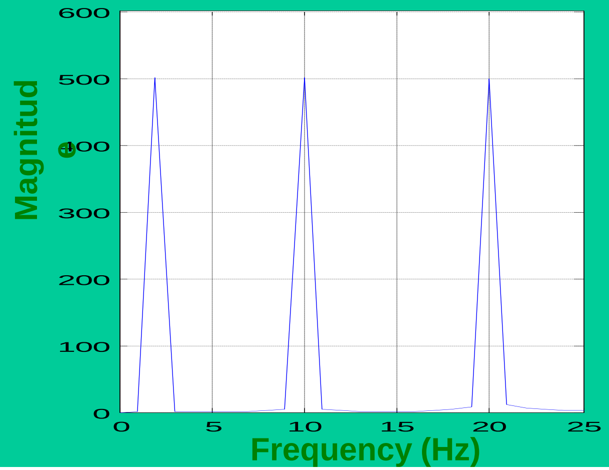
Detecting sharp changes in spectral character

STATIONARITY OF SIGNAL

2 Hz + 10 Hz + 20Hz

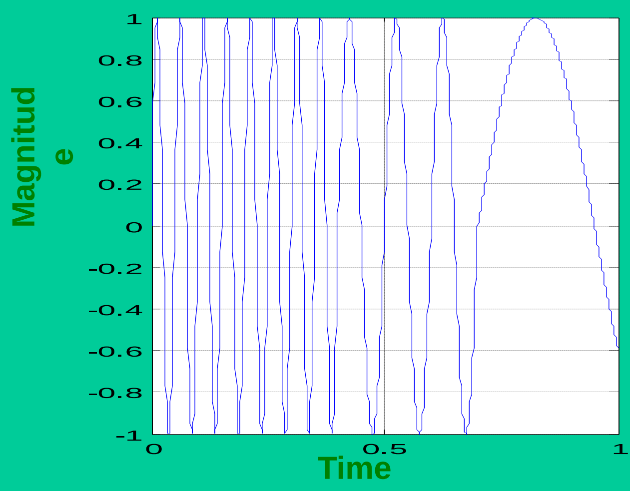
Stationary

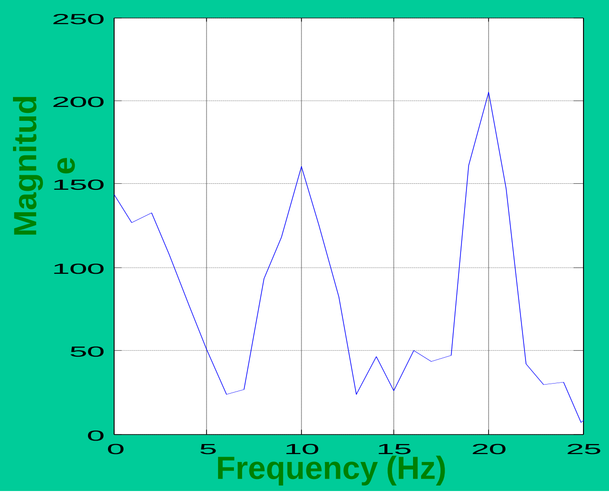




0.0-0.4: 2 Hz + 0.4-0.7: 10 Hz + 0.7-1.0: 20Hz

Non-Stationary



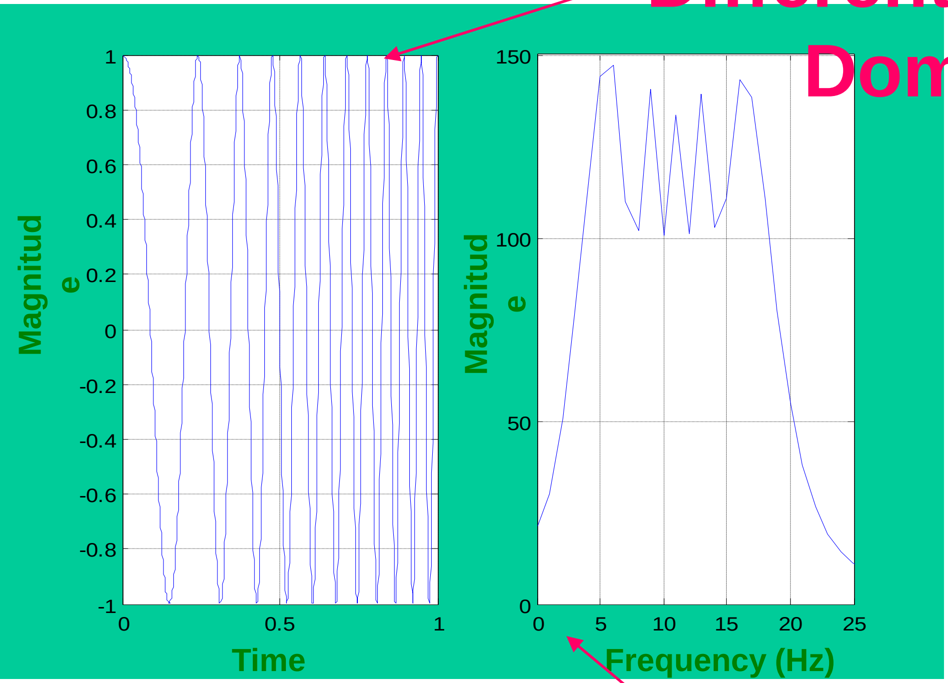


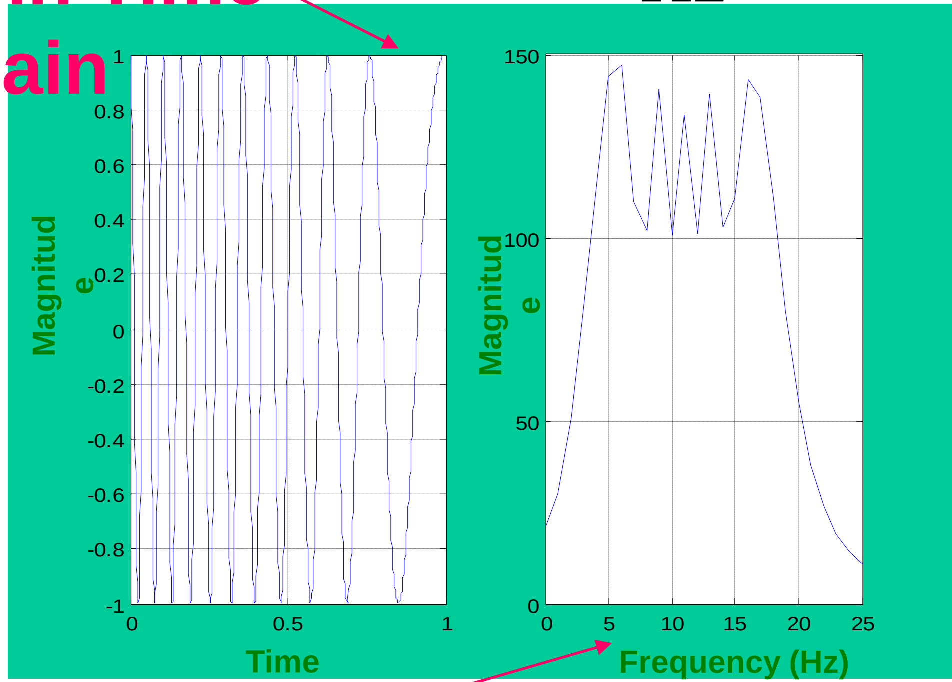
CHIRP SIGNALS

Frequency: 2 Hz to 20 Hz

Frequency: 20 Hz to Hz

Different in Time Domain

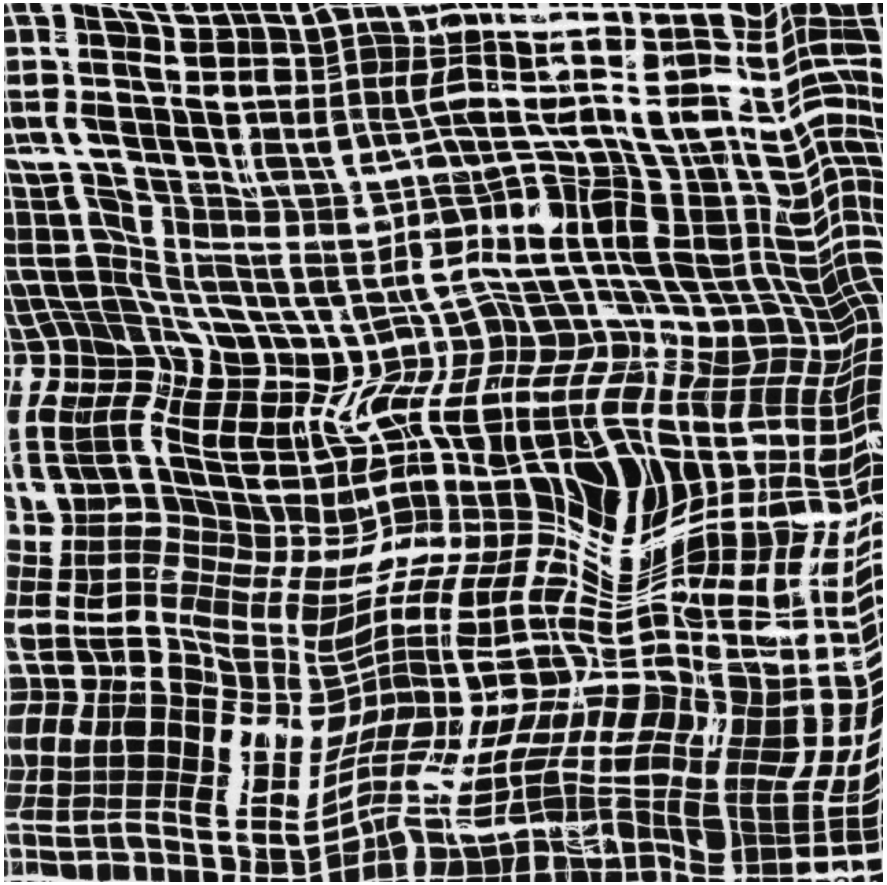


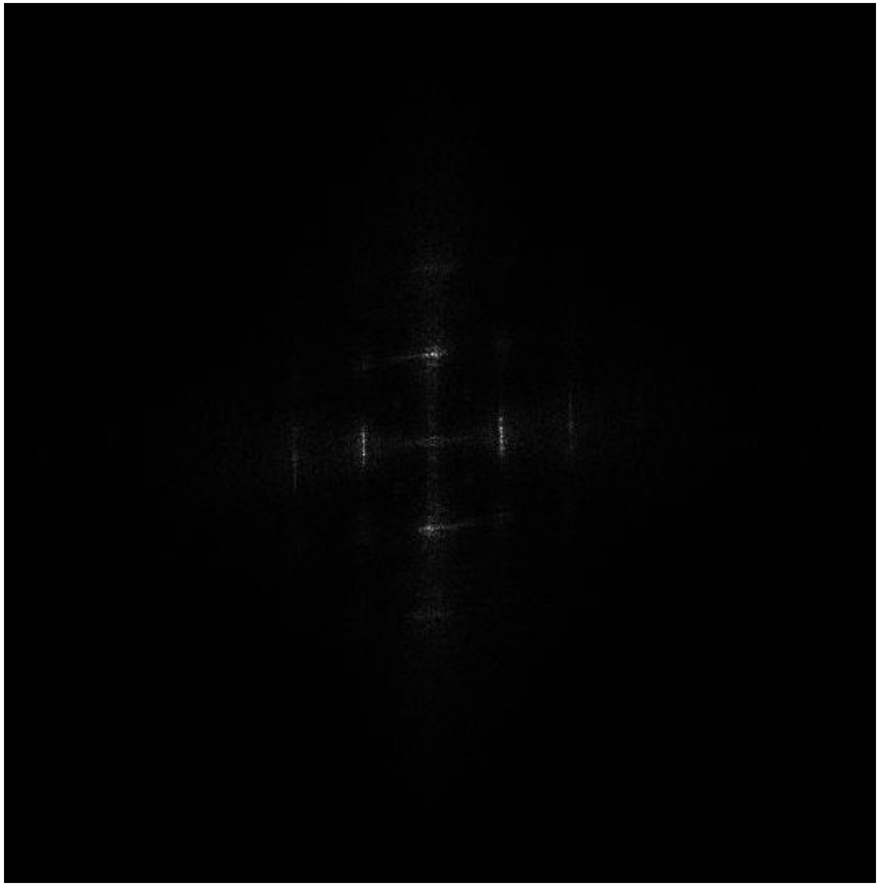


Same in Frequency Domain

At what time the frequency components occur? FT can not tell!

Shortcomings of Time and Fourier Domains





NOTHING MORE, NOTHING LESS

- **FT Only Gives what Frequency Components Exist in the Signal**
- **The Time and Frequency Information can not be Seen at the Same Time**
- **Time-frequency Representation of the Signal is Needed**

Most of Transportation Signals are Non-stationary.

(We need to know whether and also when an incident was happened.)

ONE EARLIER SOLUTION: SHORT-TIME FOURIER TRANSFORM (STFT)

SHORTCOMINGS OF THE FT

Sinusoids and exponentials

- **Stretch into infinity in time, → no time localization**
- **Instantaneous in frequency, → perfect spectral localization**
- *Global* **analysis does not allow analysis of non-stationary signals**

Need a *local* **analysis scheme for a time-frequency representation (TFR) of nonstationary signals**

- **Windowed F.T. or Short Time F.T. (STFT) : Segmenting the signal into narrow time intervals, narrow enough to be considered stationary, and then take the Fourier transform of each segment, Gabor 1946.**
- **Followed by other TFRs, which differed from each other by the selection of the windowing function**

SHORT TIME FOURIER TRANSFORM(STFT)

1. **Choose a window function of finite length**
2. **Place the window on top of the signal at t=0**
3. **Truncate the signal using this window**
4. **Compute the FT of the truncated signal, save.**
5. **Incrementally slide the window to the right**
6. **Go to step 3, until window reaches the end of the signal**

- **For each time location where the window is centered, we obtain a different FT**
  - **Hence, each FT provides the spectral information of a separate time-slice of the signal, providing simultaneous time and frequency information**

STFT

Time parameter

Frequency parameter

Signal to be analyzed

FT Kernel (basis function)

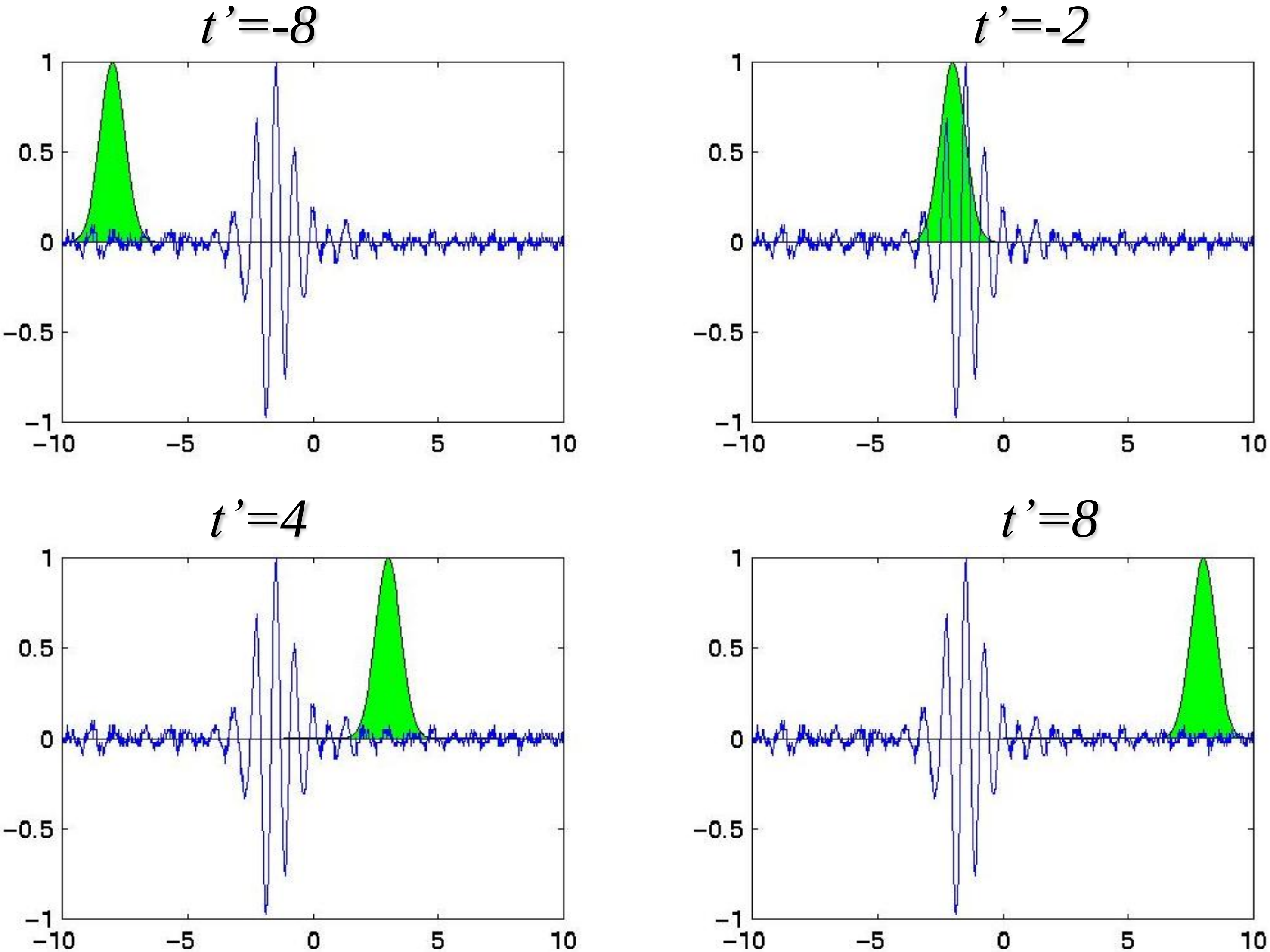
$$STFT_X^{\omega}(t',\omega)=\int_t \left[x(t)\cdot W(t-t')\right]\cdot e^{-j\omega t}dt$$

STFT of signal x(t): Computed for each window centered at t=t'

Windowing function

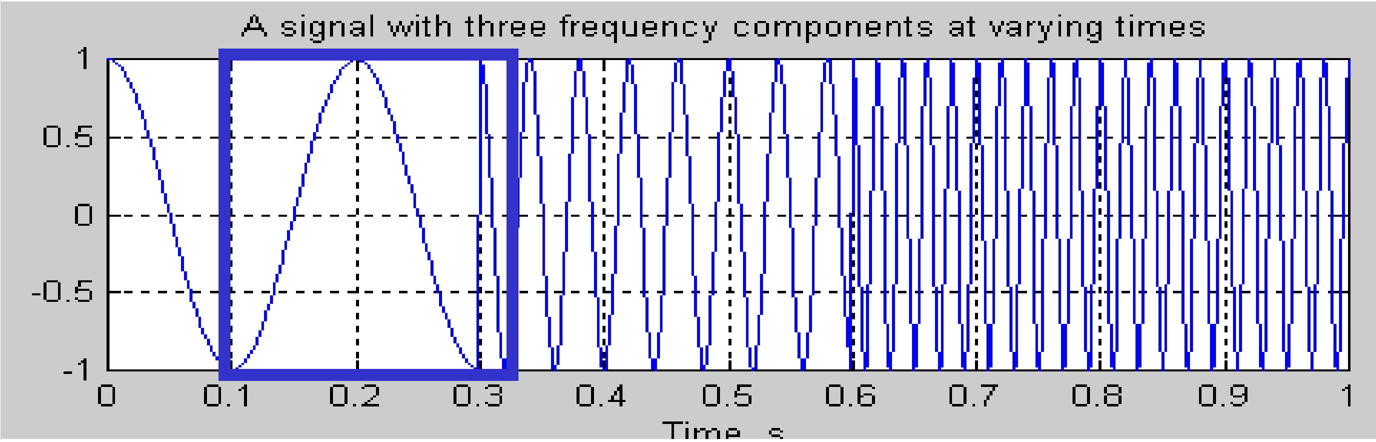
Windowing function centered at t=t'

STFT



Short Time (Windowed) Fourier Transform (STFT)

- For each time location where the window is centered, we obtain a different FT
  - Each FT provides the spectral information of a separate time-slice of the signal, providing simultaneous time and frequency information

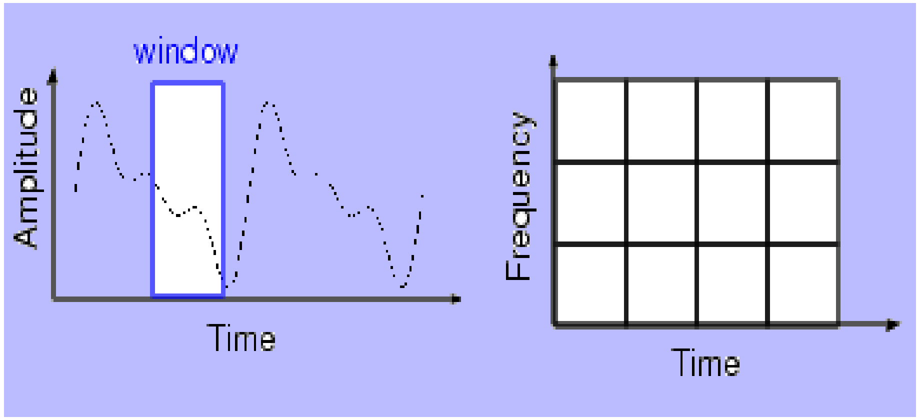


SFORT TIME FOURIER TRANSFORM (STFT)

- **Dennis Gabor (1946) Used STFT**
  - **To analyze only a small section of the signal at a time -- a technique called *Windowing the Signal*.**
- **The Segment of Signal is Assumed *Stationary***
- **A 3D transform**

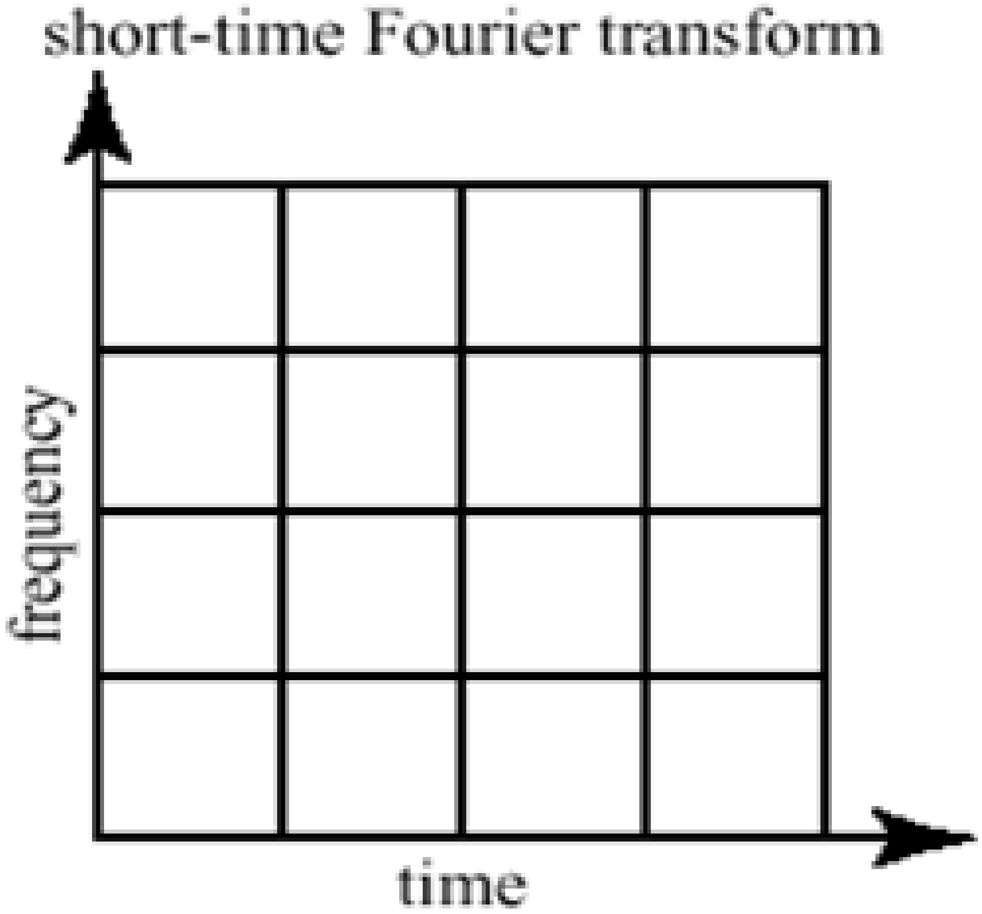
$$\mathrm{STFT}_X^{(\omega)}(t', f) = \int_t \left[x(t) \bullet \omega^*(t - t')\right] \bullet e^{-j2\pi ft}\, dt$$

$\omega(t)$: the window function

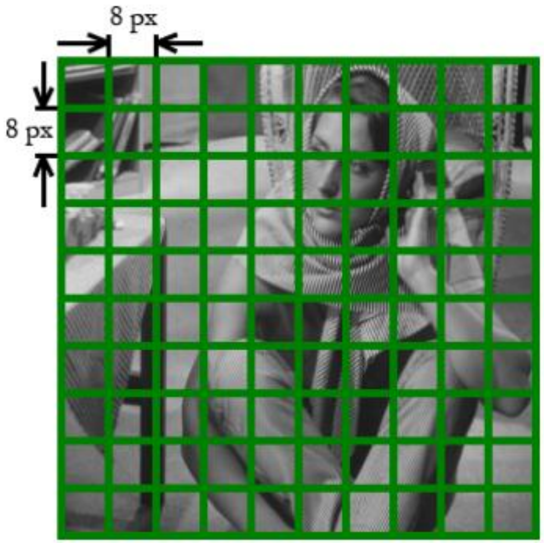


A function of time and frequency

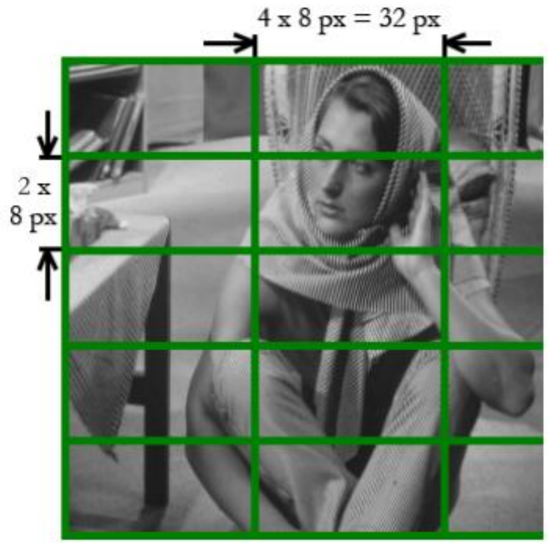
STFT Interpretation



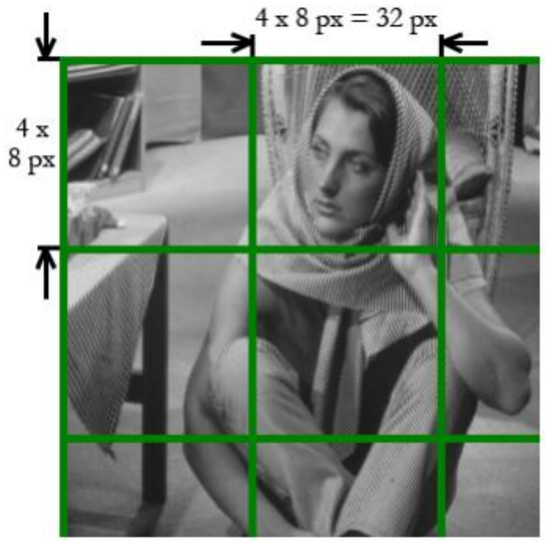
STFT of Images



Barbara image with 8x8 blocks (1st kernel)



Barbara image split into macroblocks. Each macroblock contains 8 blocks (2nd kernel)



Barbara image split into macroblocks. Each macroblock contains 16 blocks (short kernel)

STFT

- STFT provides the time information by computing a different FTs for consecutive time intervals, and then putting them together
  - Time-Frequency Representation (TFR)
  - Maps 1-D time domain signals to 2-D time-frequency signals
- Consecutive time intervals of the signal are obtained by truncating the signal using a sliding windowing function
- How to choose the windowing function?
  - What shape? Rectangular, Gaussian, Elliptic…?
  - How wide?
    - Wider window require less time steps → low time resolution
    - Also, window should be narrow enough to make sure that the portion of the signal falling within the window is stationary
    - Can we choose an arbitrarily narrow window…?
- *Wide analysis window→ poor time resolution, good frequency resolution*
- *Narrow analysis window→good time resolution, poor frequency resolution*
- *Once the window is chosen, the resolution is set for both time and frequency.*

SELECTION OF STFT WINDOW

$$STFT_X^{\omega}(t',\omega) = \int_t \left[x(t)\cdot W(t-t')\right]\cdot e^{-j\omega t}dt$$

Two extreme cases:

- ***W(t)* infinitely long:** $W(t) = 1$ ➔ **STFT turns into FT, providing excellent frequency information (good frequency resolution), but no time information**
- ***W(t)* infinitely short:**

$$STFT_X^{\omega}(t',\omega) = \int_t \left[x(t)\cdot \delta(t-t')\right]\cdot e^{-j\omega t}dt = x(t')\cdot e^{-j\omega t'}$$

$W(t) = \delta(t)$ ➔ **STFT then gives the time signal back, with a phase factor. Excellent time information (good time resolution), but no frequency information**

Wide analysis window→ poor time resolution, good frequency resolution

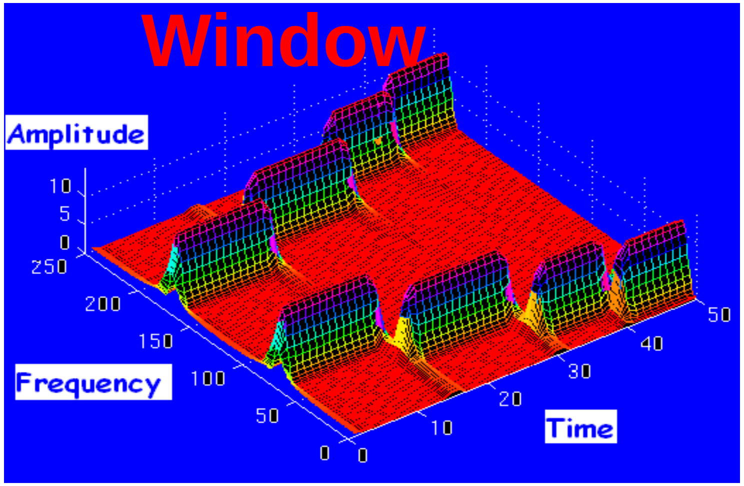
Narrow analysis window→good time resolution, poor frequency resolution

Once the window is chosen, the resolution is set for both time and frequency.

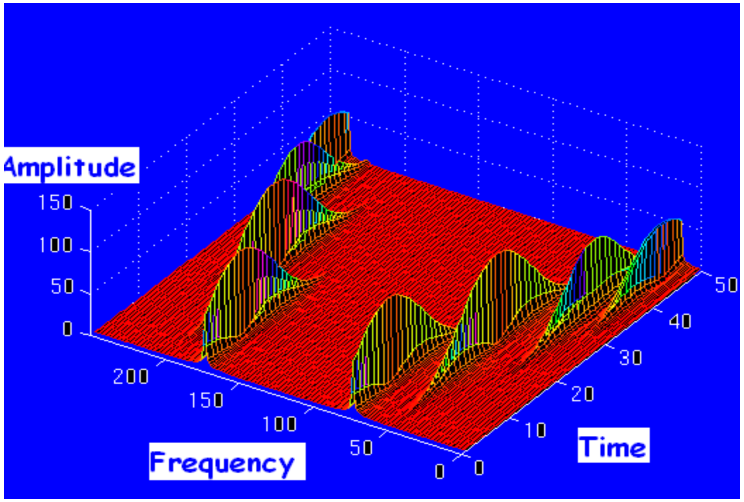
DRAWBACKS OF STFT

- **Unchanged Window**
- **Dilemma of Resolution**
  - **Narrow window -> poor frequency resolution**
  - **Wide window -> poor time resolution**
- **Heisenberg Uncertainty Principle**
  - **Cannot know what frequency exists at what time intervals**

Via Narrow Window



Via Wide



$$\Delta t \cdot \Delta f \geq \frac{1}{4\pi}$$

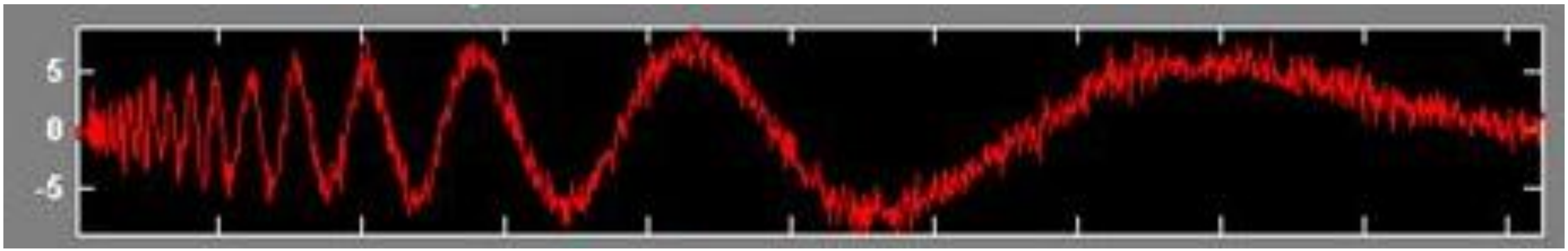
Time resolution: How well two spikes in time can be separated from each other in the transform domain

Frequency resolution: How well two spectral components can be separated from each other in the transform domain

Both time and frequency resolutions cannot be arbitrarily high!!!

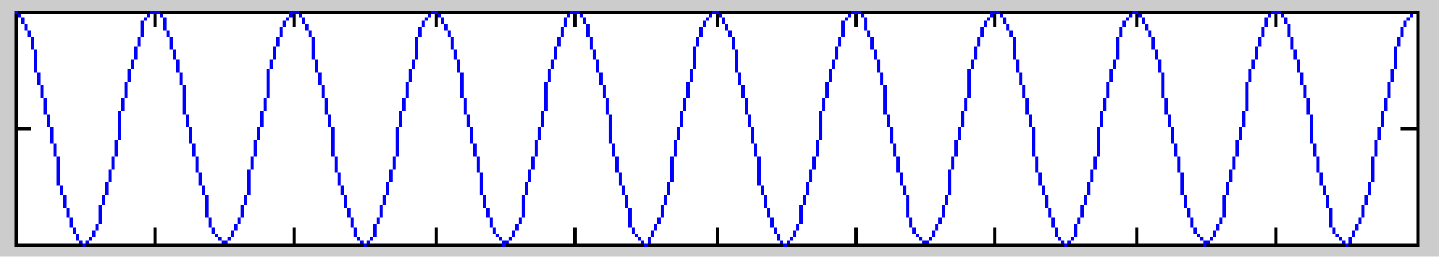
➔ ➔We cannot precisely know at what time instance a frequency component is located. We can only know what *interval of frequencies* are present in which *time intervals*

Wavelet Motivation



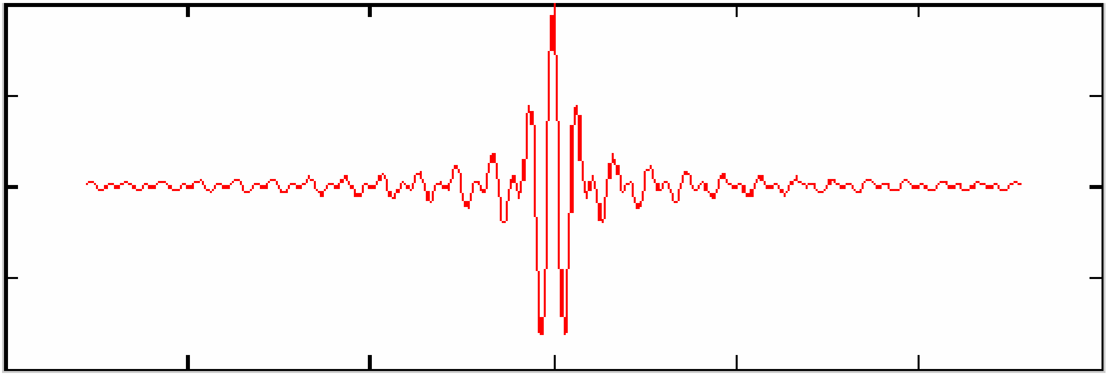
Some signals obviously have spectral characteristics that vary with time

Fourier Analysis is based on an indefinitely long cosine wave of a specific frequency



time, t

Wavelet Analysis is based on an short duration wavelet of a specific *center frequency*



time, t

THE WAVELET TRANSFORM

- **Overcomes the preset resolution problem of the STFT by using a variable length window**
- **Analysis windows of different lengths are used for different frequencies:**
  - **Analysis of high frequencies→ Use narrower windows for better time resolution**
  - **Analysis of low frequencies → Use wider windows for better frequency resolution**
- **This works well, if the signal to be analyzed mainly consists of slowly varying characteristics with occasional short high frequency bursts.**
- **The function used to window the signal is called** *the wavelet*

THE WAVELET TRANSFORM

Translation parameter, measure of time

Scale parameter, measure of frequency

A normalization constant

Signal to be analyzed

$$CWT_x^{\psi}(\tau, s) = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int_t x(t)\, \psi^*\left(\frac{t-\tau}{s}\right) dt$$

Continuous wavelet transform of the signal *x(t)* using the analysis wavelet ψ(.)

The mother wavelet. All kernels are obtained by translating (shifting) and/or scaling the mother wavelet

Scale = 1/frequency

Wavelet Transform

$$\gamma(s,\tau) = \int f(t)\psi^{*}_{s,\tau}(t)dt$$

Inverse Wavelet Transform

$$f(t) = \int\int \gamma(s,\tau)\psi_{s,\tau}(t)d\tau ds$$

All wavelet derived from *mother wavelet*

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

Wavelet Transform

I'm going to ignore the complex conjugate from now on, assuming that we're using real wavelets

time-series

$$\gamma(s,\tau) = \int f(t)\psi^{*}_{s,\tau}(t)\,dt$$

coefficient of wavelet with
scale, s and time, τ

complex conjugate of wavelet with
scale, s and time, τ

Wavelet

normalization

shift in time

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\,\psi\left(\frac{t-\tau}{s}\right)$$

change in scale:
big s means long
wavelength

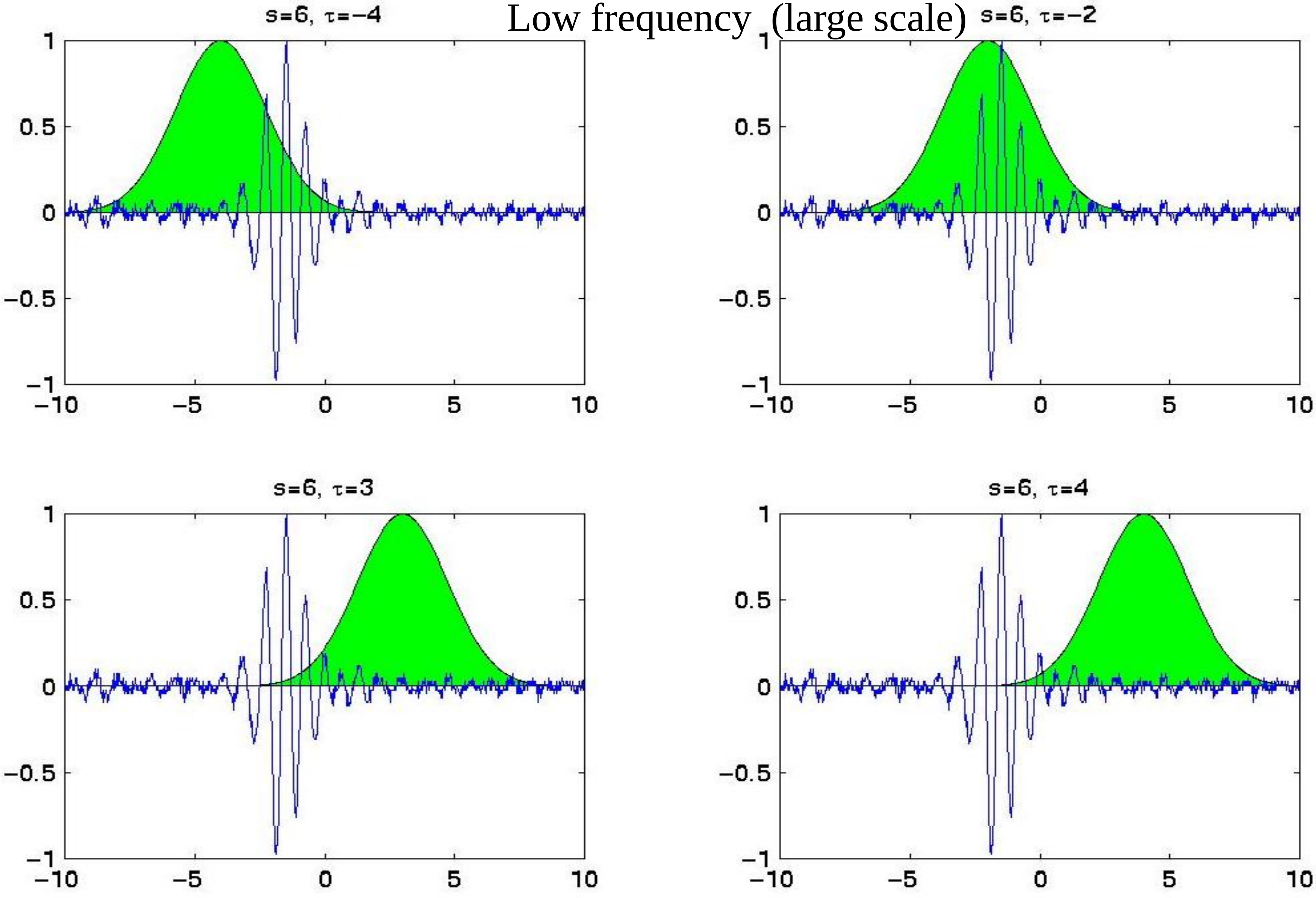
wavelet with
scale, s and time, τ

Mother wavelet

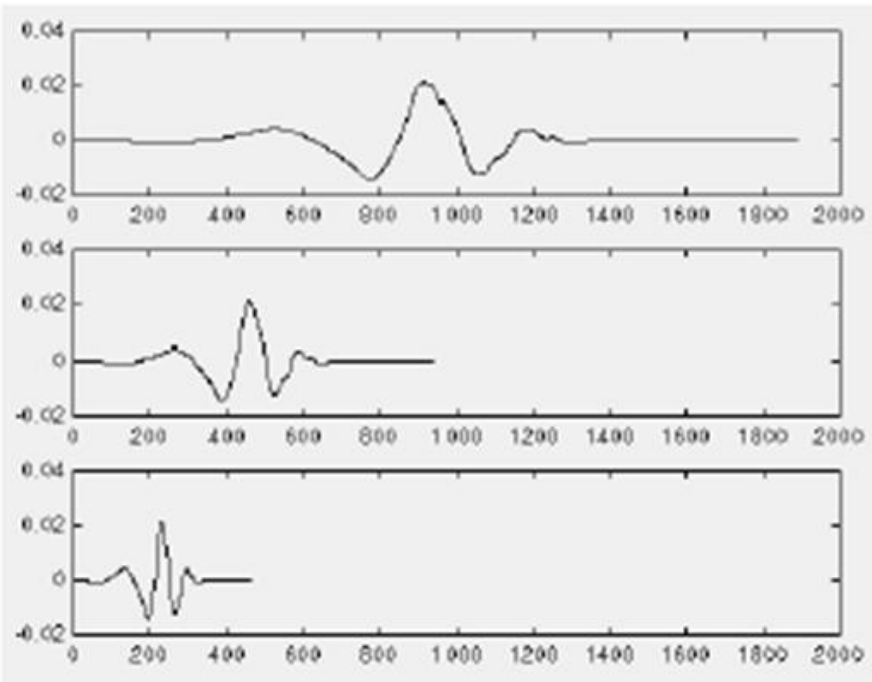
WT at Work

$$CWT_x^{\psi}(\tau,s)=\Psi_x^{\psi}(\tau,s)=\frac{1}{\sqrt{|s|}}\int_t x(t)\psi^*\left(\frac{t-\tau}{s}\right)dt$$

Low frequency (large scale)



Scaling of mother wavelet and its relation to frequency



$$f(t) = \psi(t) \quad ; \quad a = 1$$

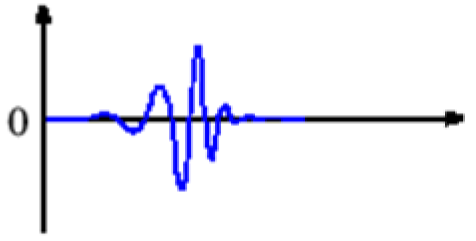
$$f(t) = \psi(2t) \quad ; \quad a = \frac{1}{2}$$

$$f(t) = \psi(4t) \quad ; \quad a = \frac{1}{4}$$

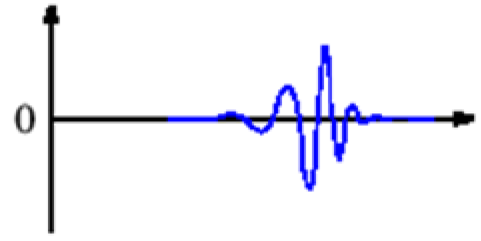
- Low scale a → Compressed wavelet → Rapidly changing details → High frequency ω

- High scale a → stretched wavelet → slowly changing details → low frequency ω

Translation of mother wavelet



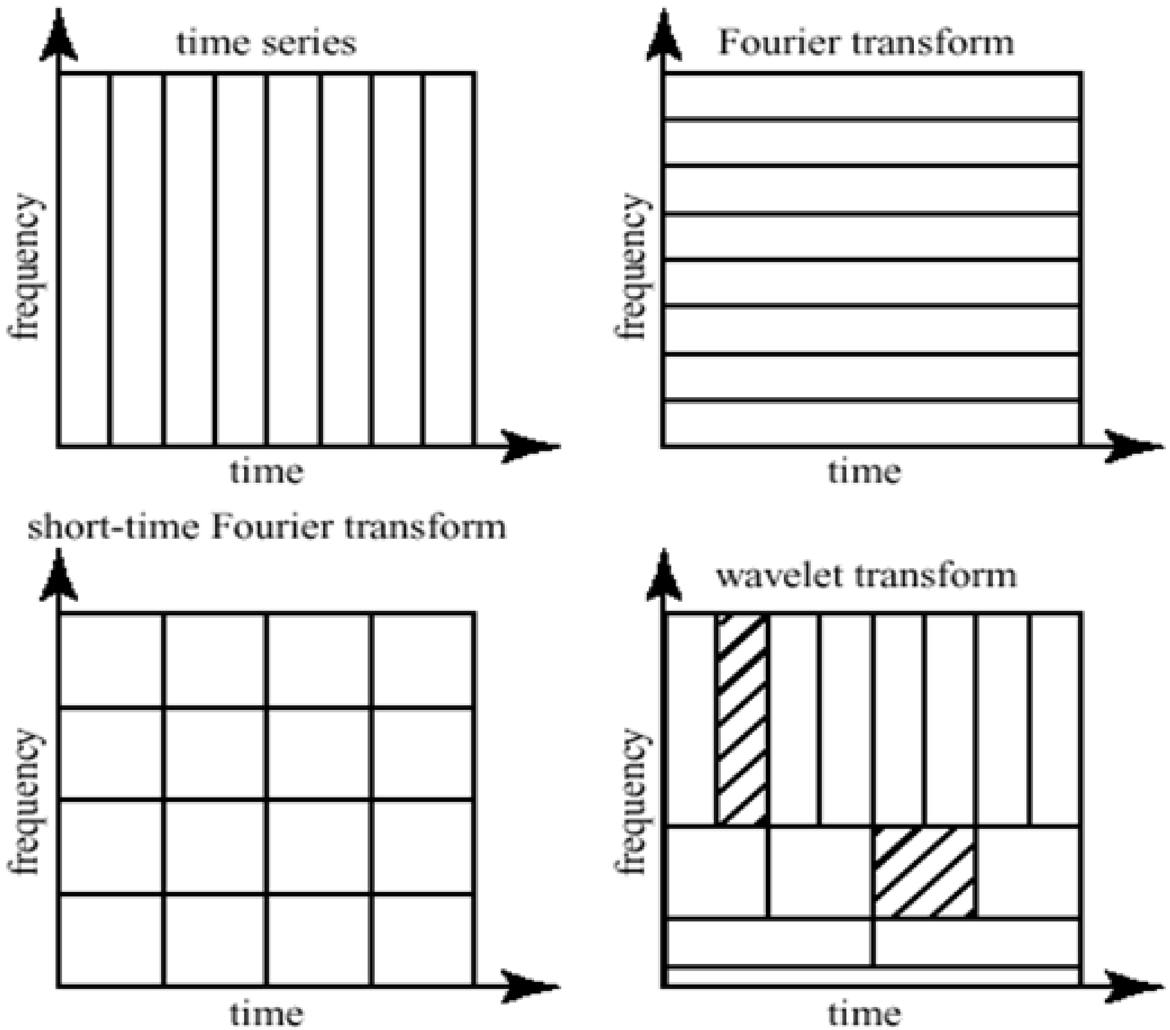
Wavelet function
$\psi(t)$



Shifted wavelet function
$\psi(t - k)$

- Place the mother wavelet at every location in the signal to calculate the frequency in the window equivalent to the size of the mother wavelet (scale ‘s’)

Time, Fourier, Windowed Fourier and Wavelet Comparisons



Inverse Wavelet Transform

$$f(t) = \int\int \gamma(s,\tau)\psi_{s,\tau}(t)\,d\tau\,ds$$

time-series

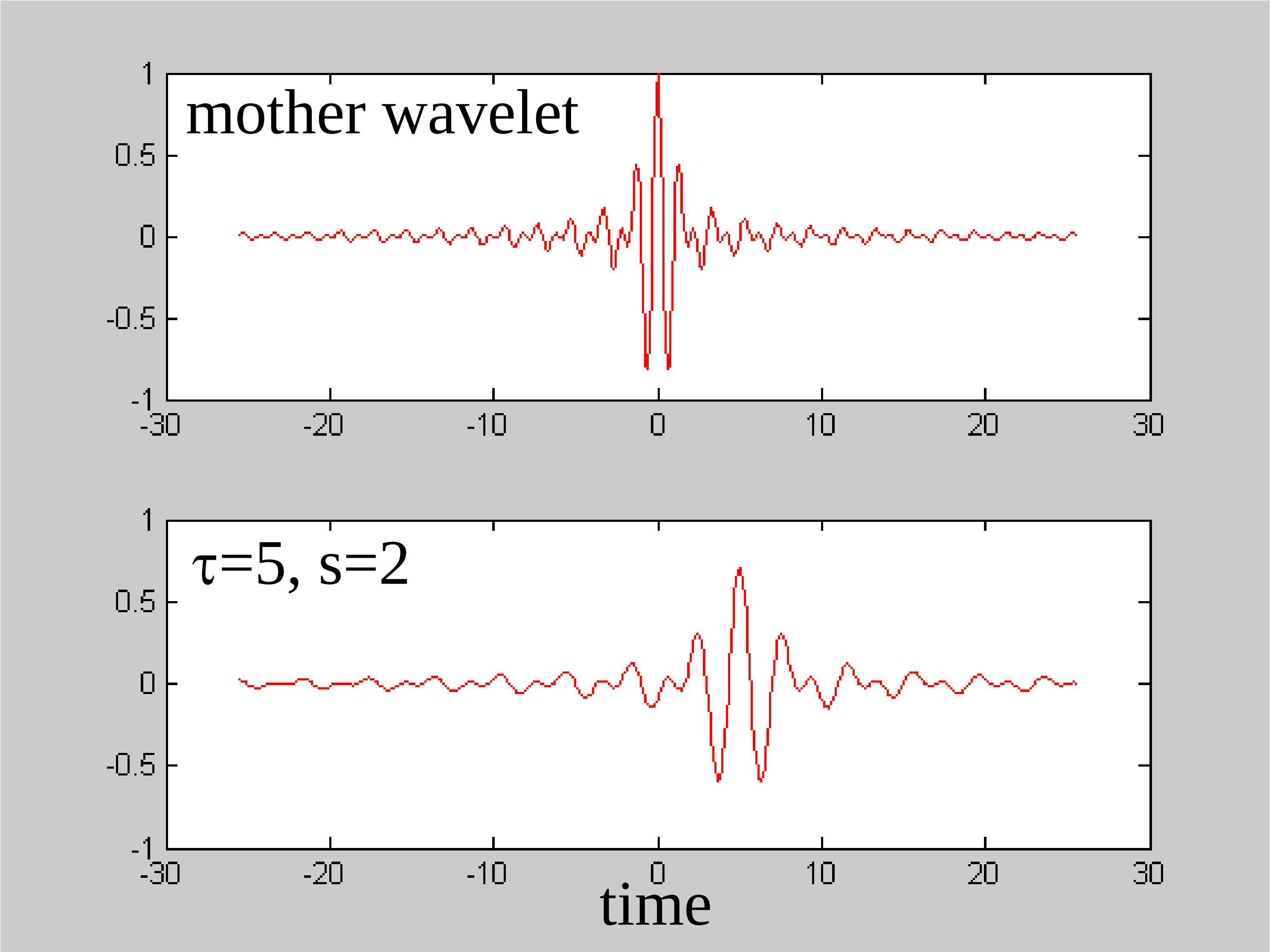
coefficients of wavelets

wavelet with scale, s and time, τ

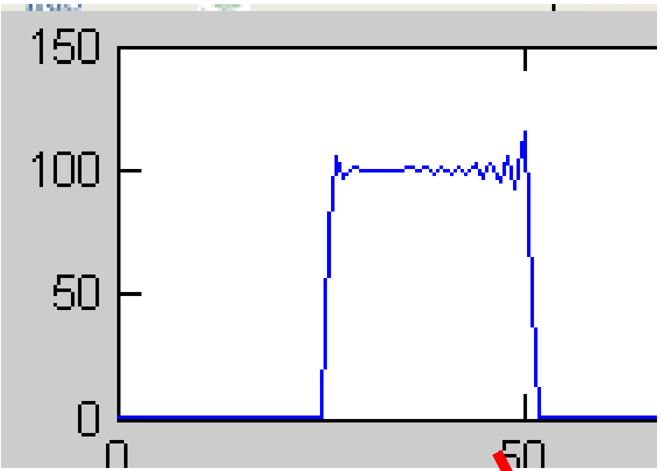
build up a time-series as sum of wavelets of different scales, s, and positions, τ

Shannon Wavelet

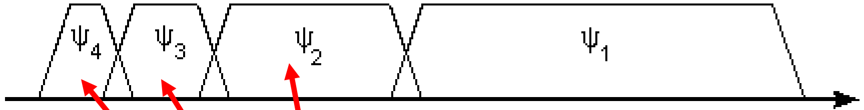
$$\Psi(t) = 2\ \text{sinc}(2t) - \text{sinc}(t)$$



Fourier spectrum of Shannon Wavelet



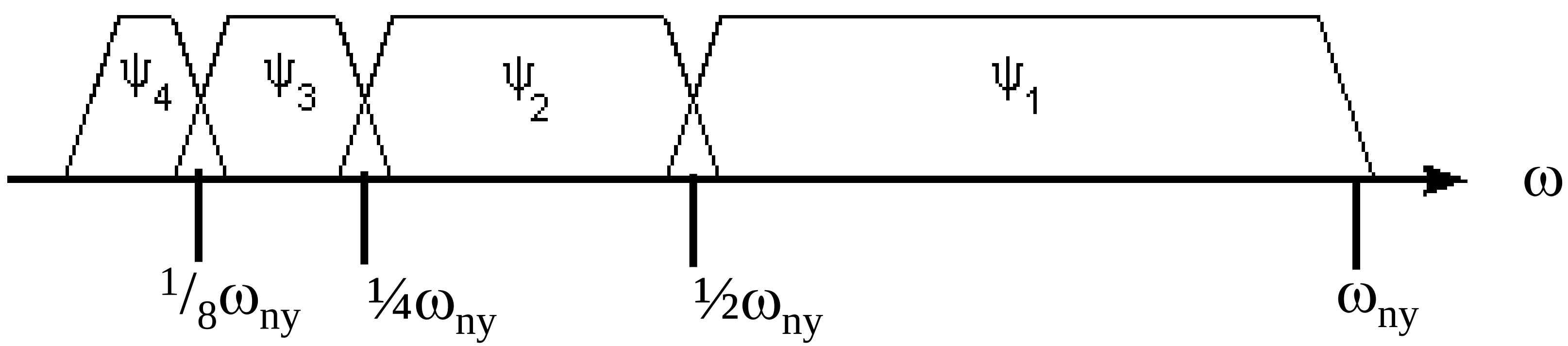
frequency, ω



ω

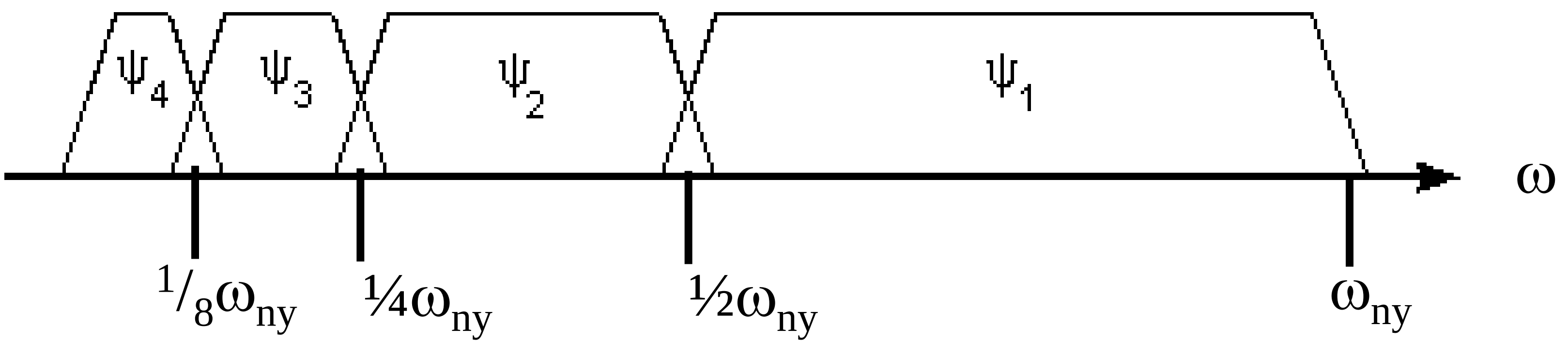
Spectrum of higher scale wavelets

The factor of two scaling means that the spectra of the wavelets divide up the frequency scale into *octaves* (frequency doubling intervals)

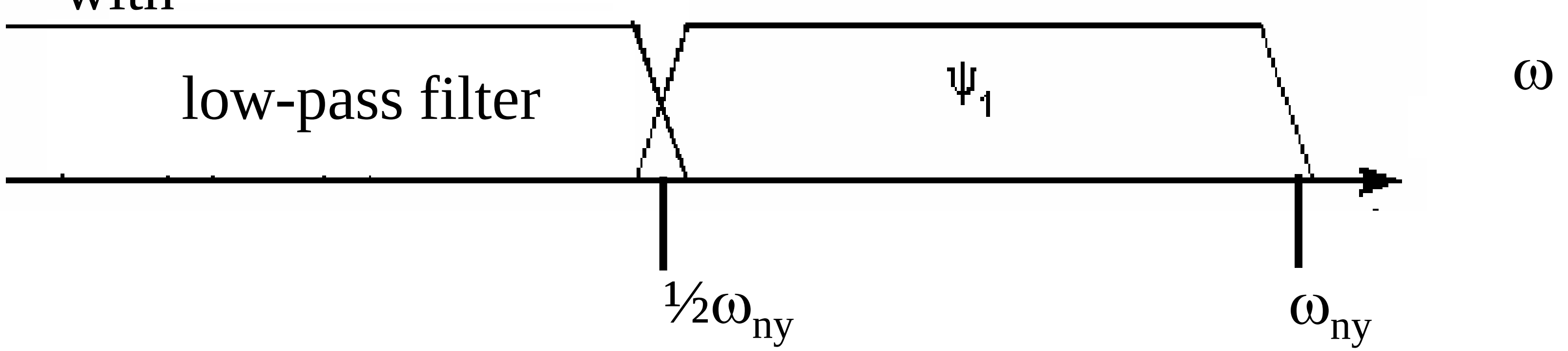


As we showed previously, the coefficients of Ψ_1 is just the band-passes filtered time-series, where Ψ_1 is the wavelet, now viewed as a bandpass filter.

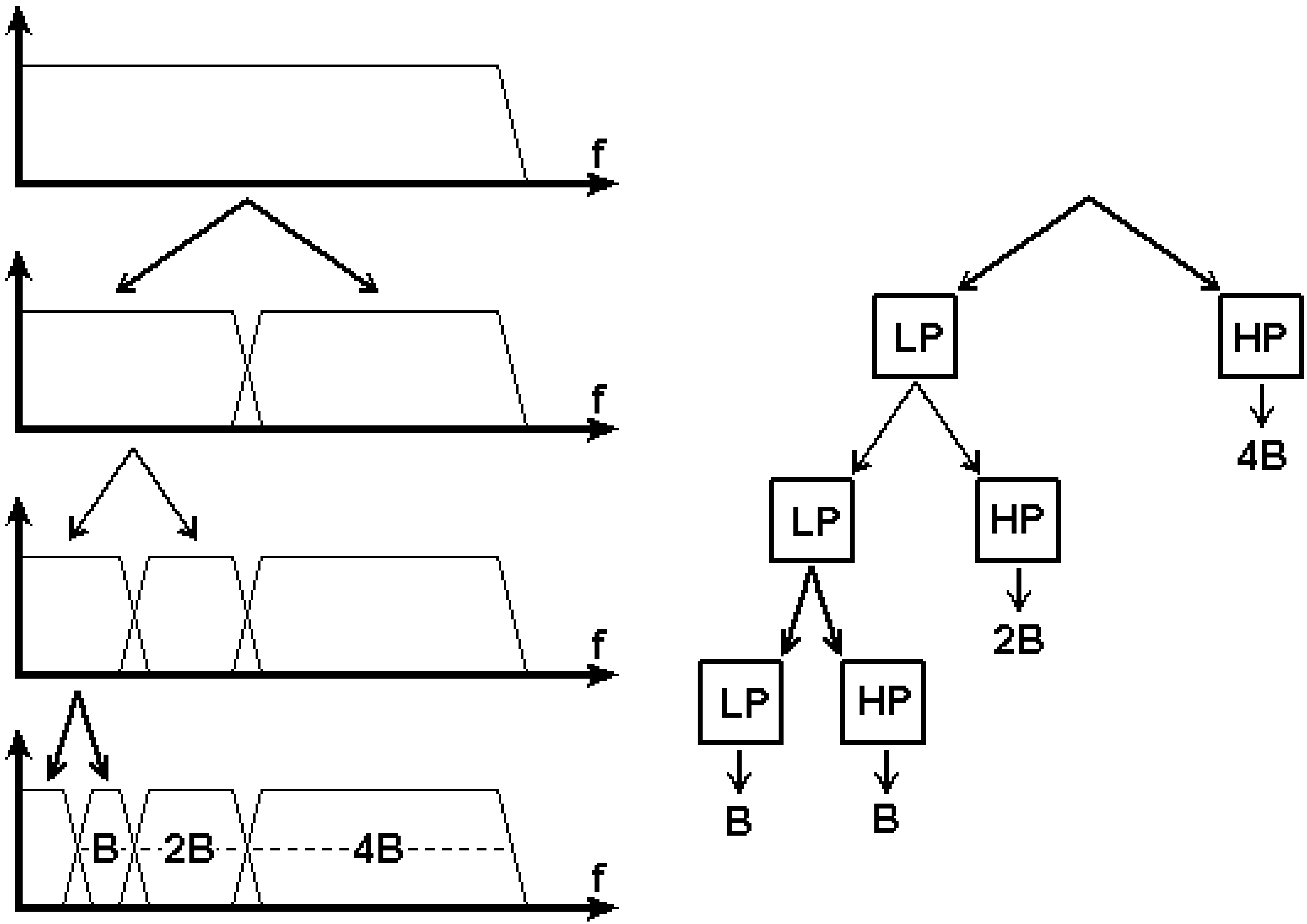
This suggests a recursion. Replace:

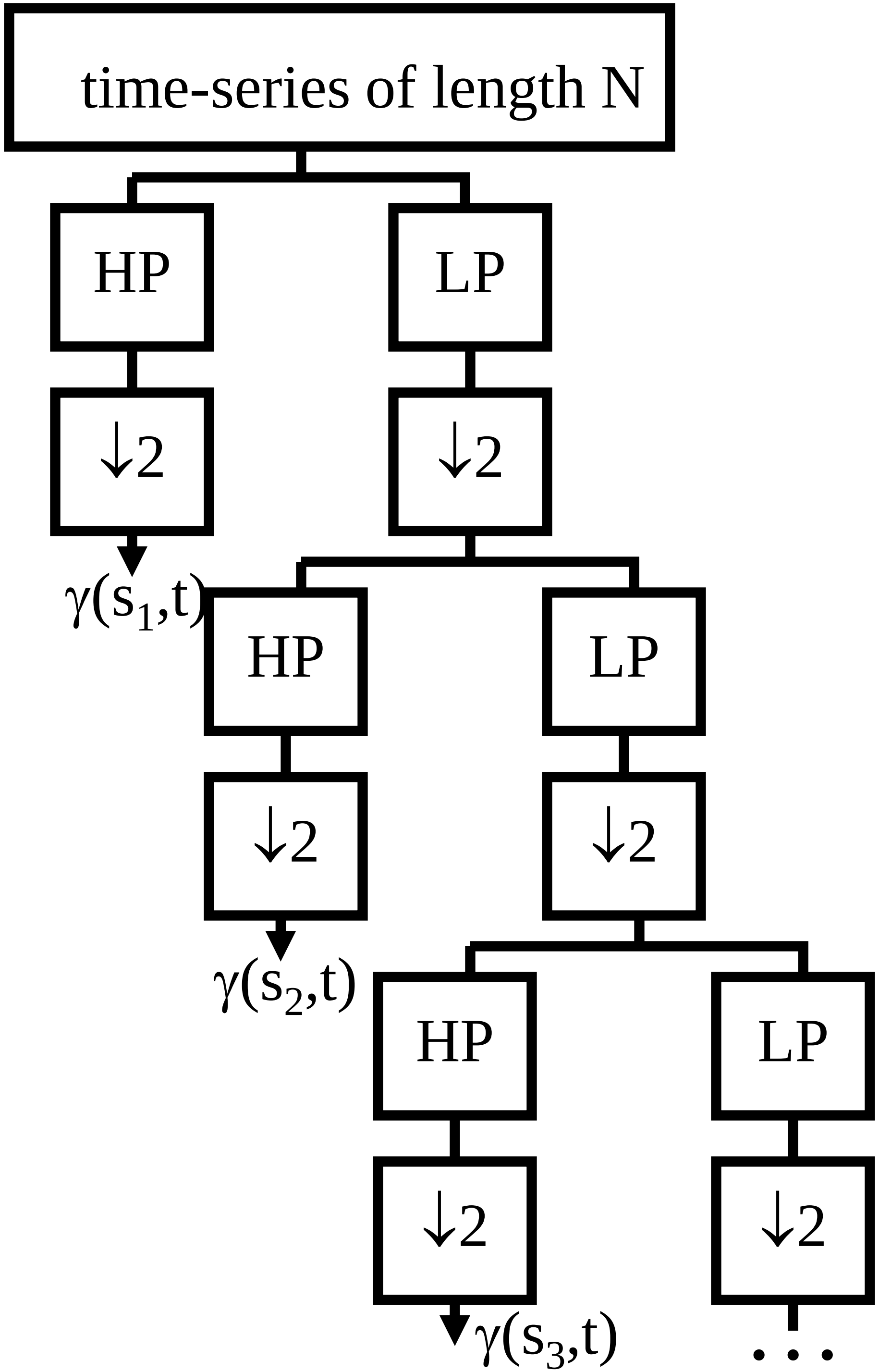


with



And then repeat the processes, recursively …





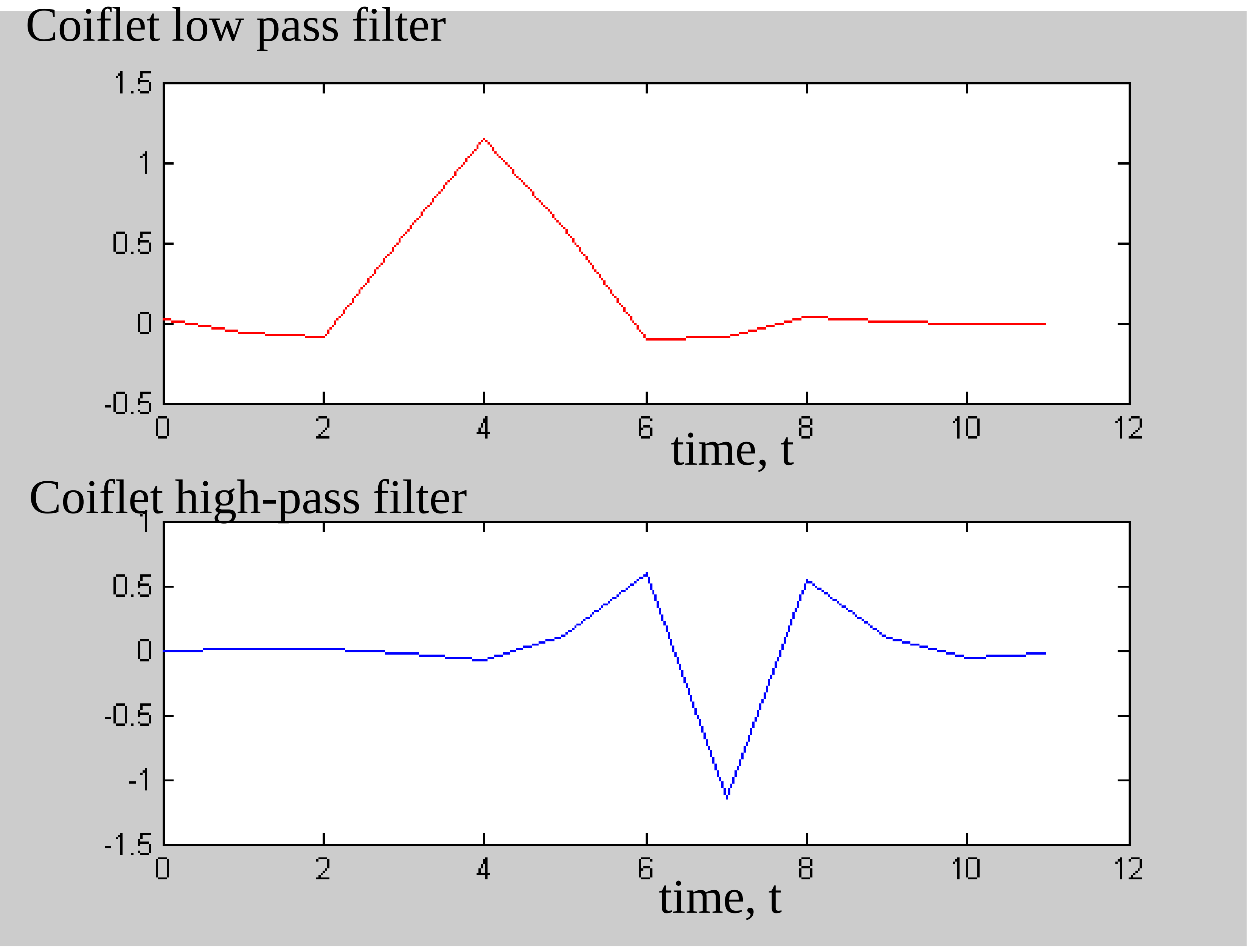
Recursion for wavelet coefficients

$\gamma(s_1,t)$: N/2 coefficients

$\gamma(s_2,t)$: N/4 coefficients

$\gamma(s_2,t)$: N/8 coefficients

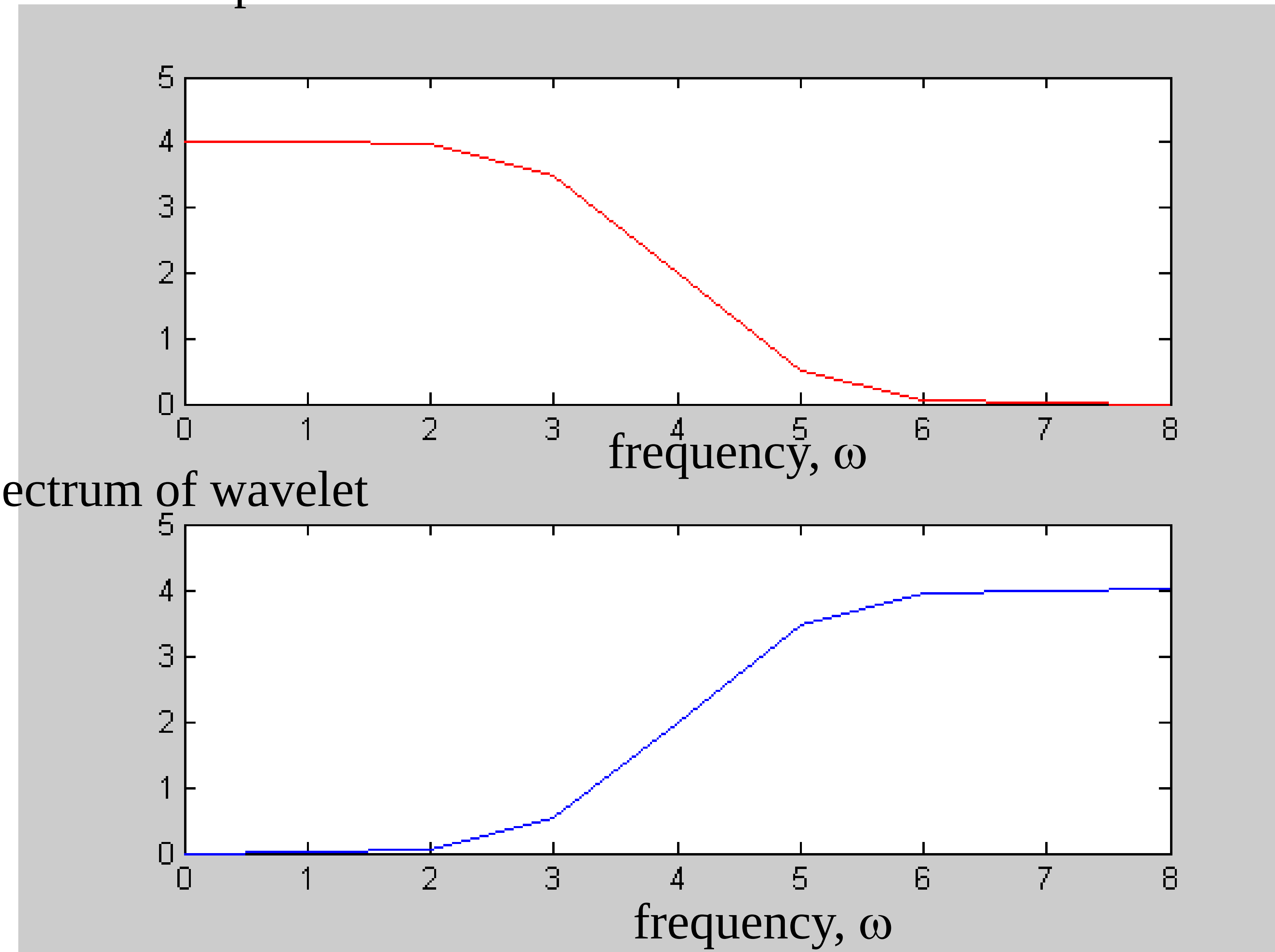
Total: N coefficients

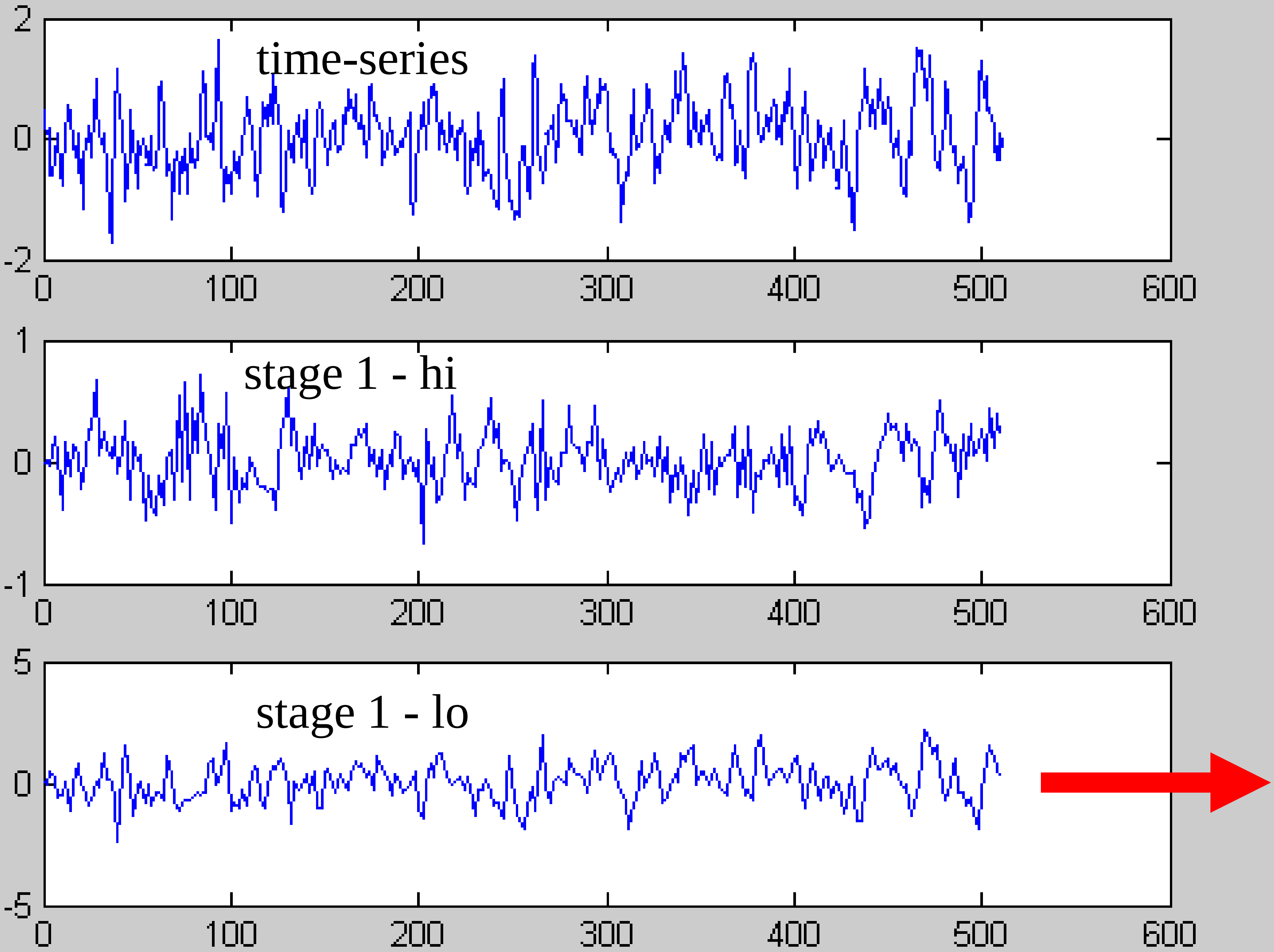


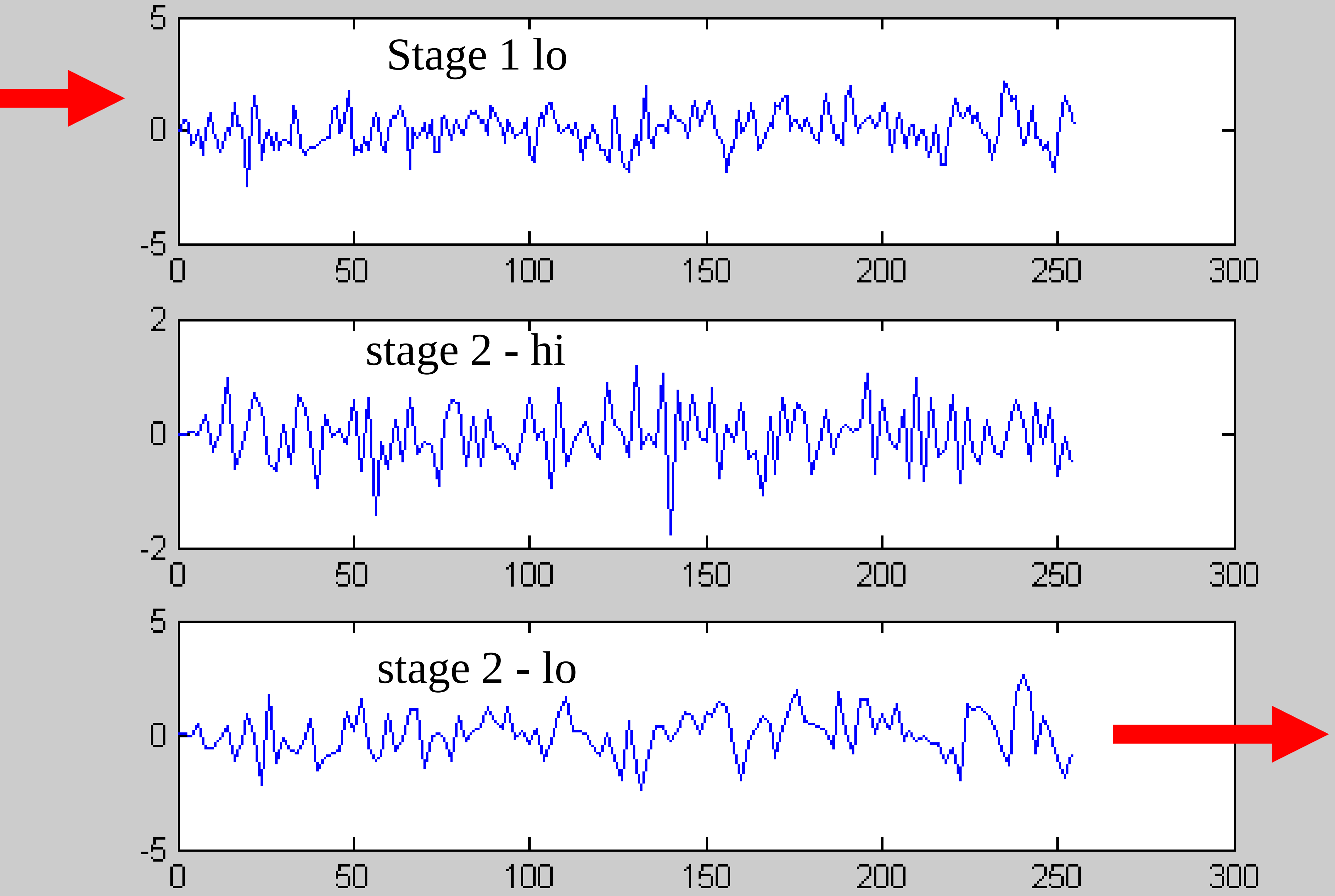
From http://en.wikipedia.org/wiki/Coiflet

Spectrum of low pass filter

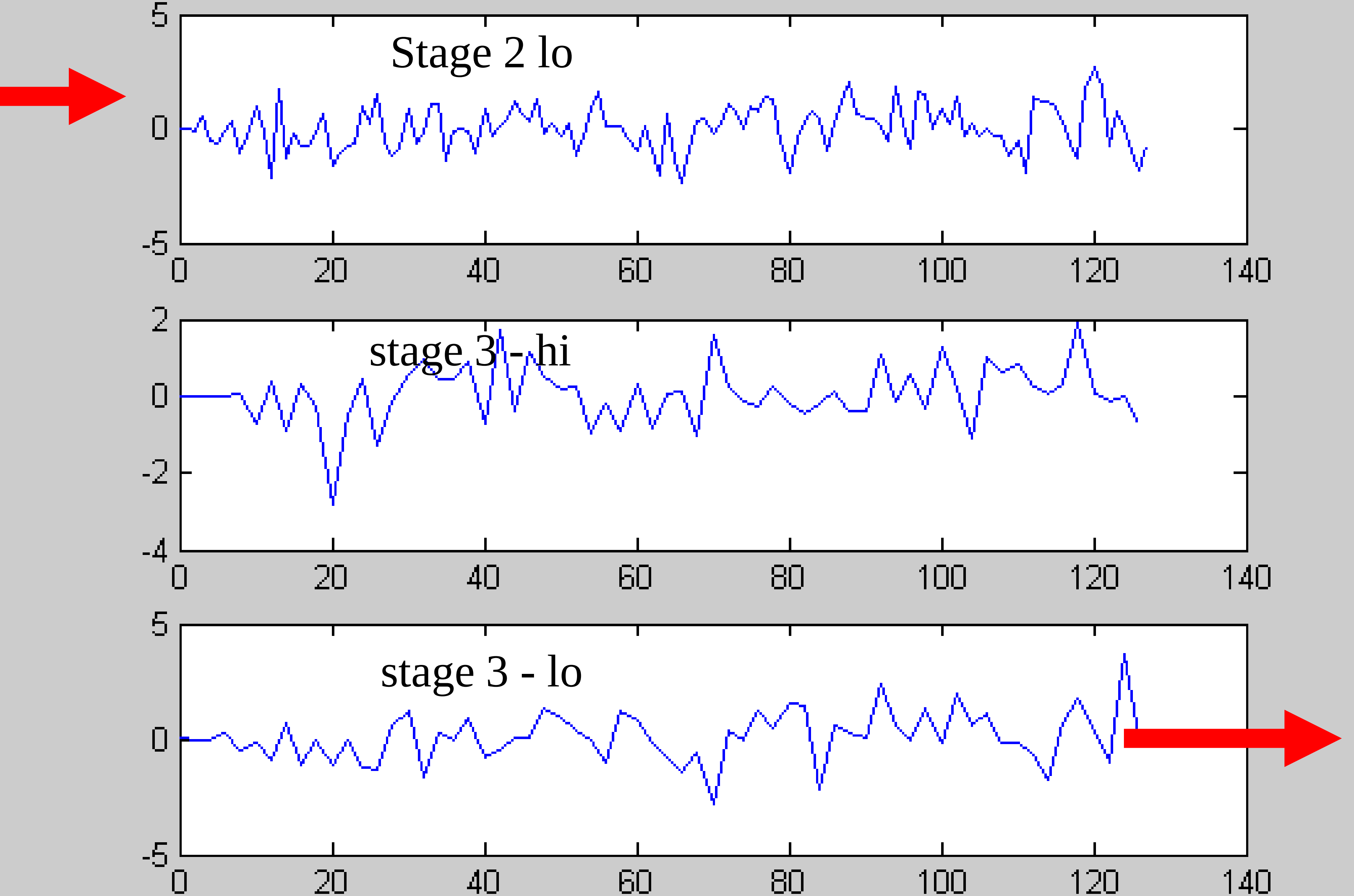
Spectrum of wavelet

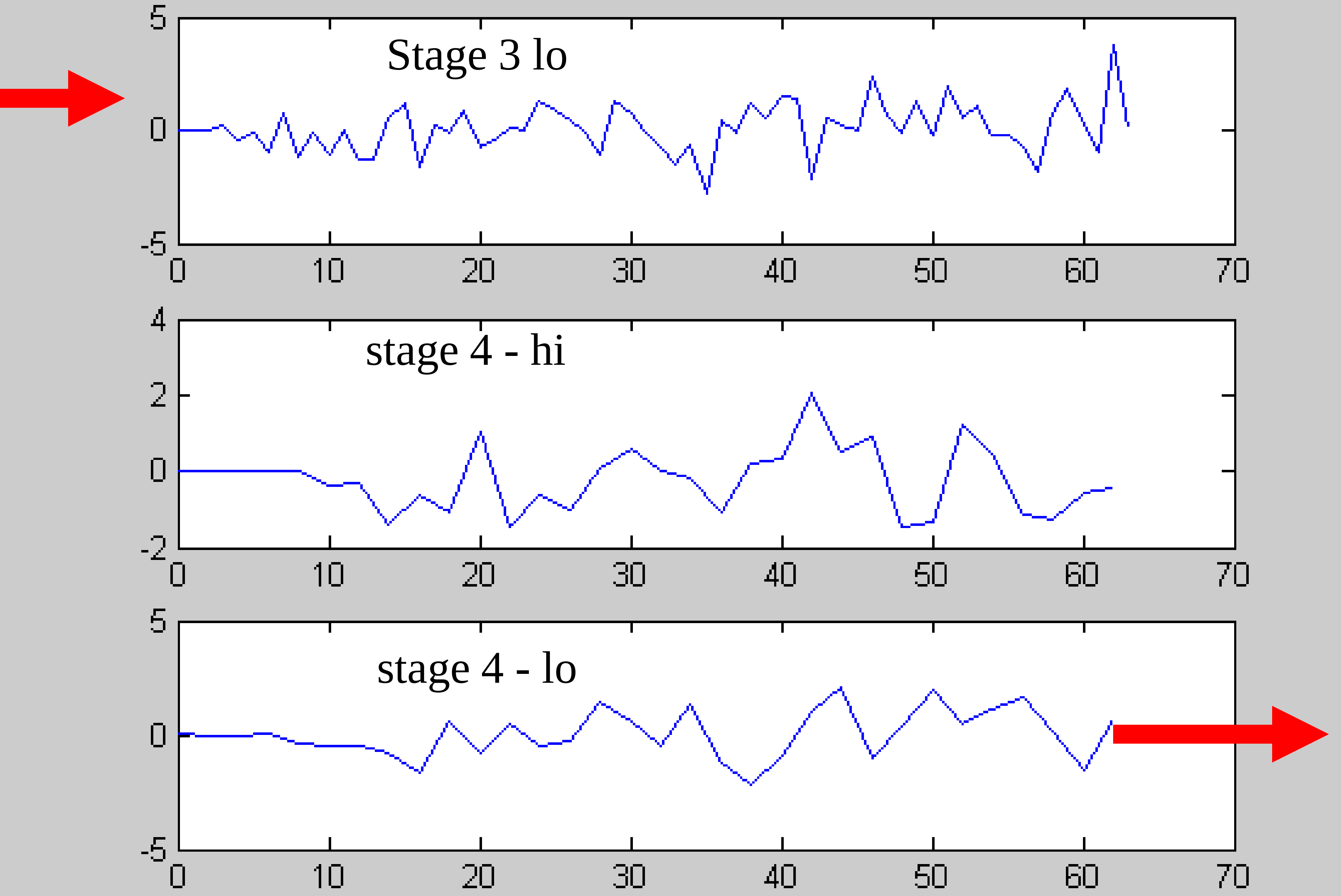



time-series
stage 1 - hi
stage 1 - lo
2
0
-2
1
0
-1
5
0
-5
0
100
200
300
400
500
600

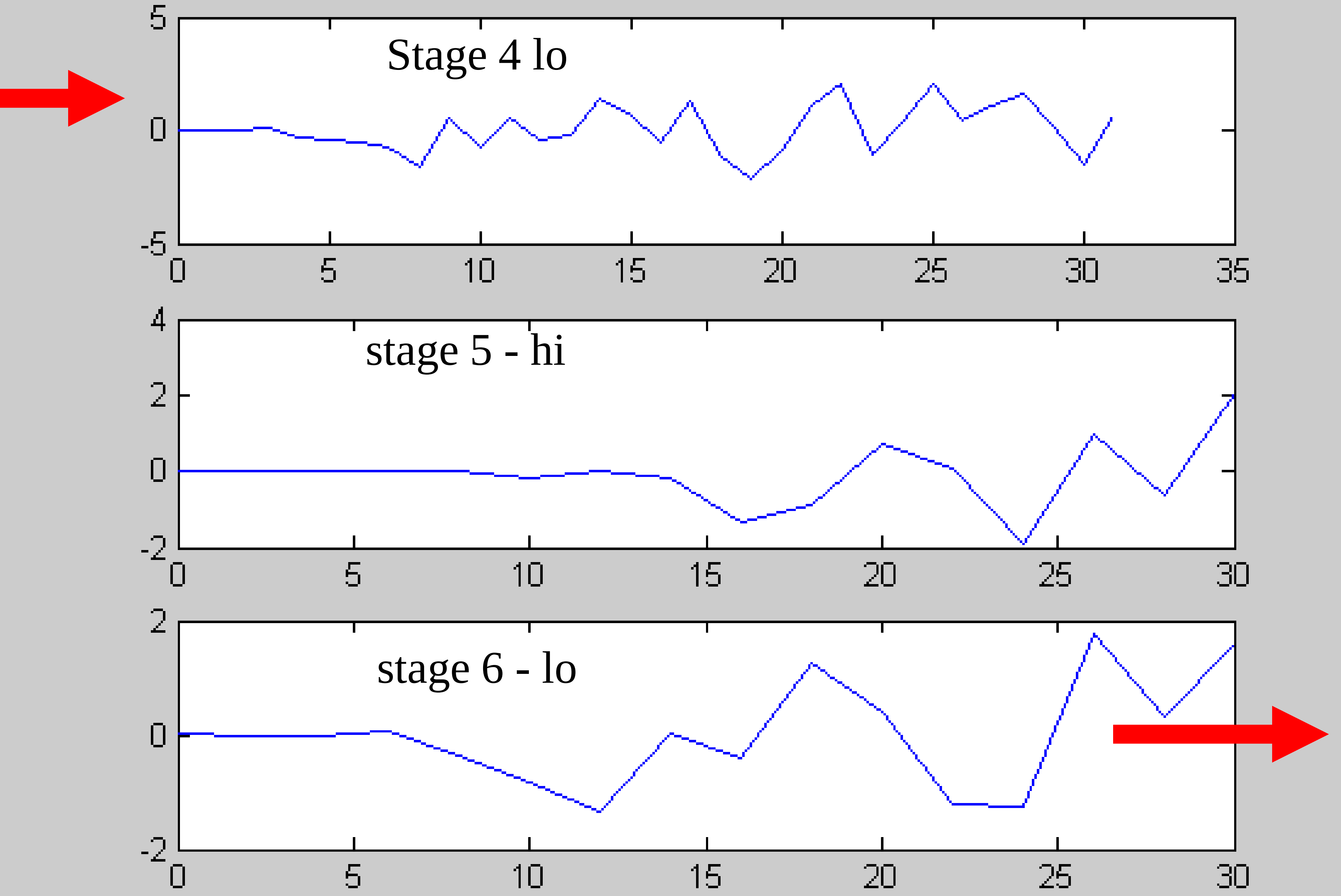


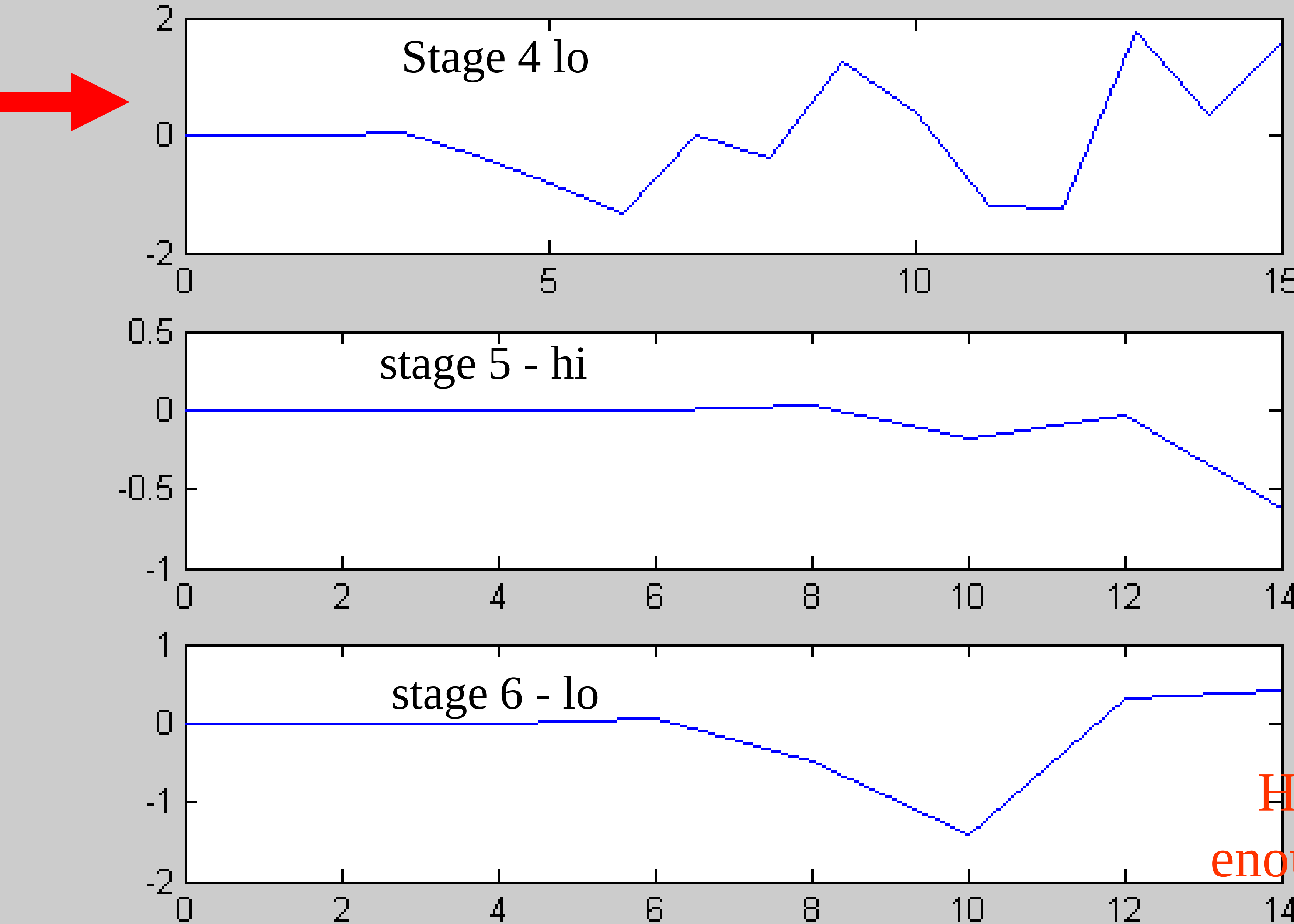
Stage 1 lo
stage 2 - hi
stage 2 - lo
5
0
-5
2
-2
0
50
100
150
200
250
300


Stage 2 lo
stage 3 - hi
stage 3 - lo
5
0
-5
2
0
-2
-4
0
20
40
60
80
100
120
140



Stage 3 lo
stage 4 - hi
stage 4 - lo
5
0
-5
4
2
0
-2
0
10
20
30
40
50
60
70

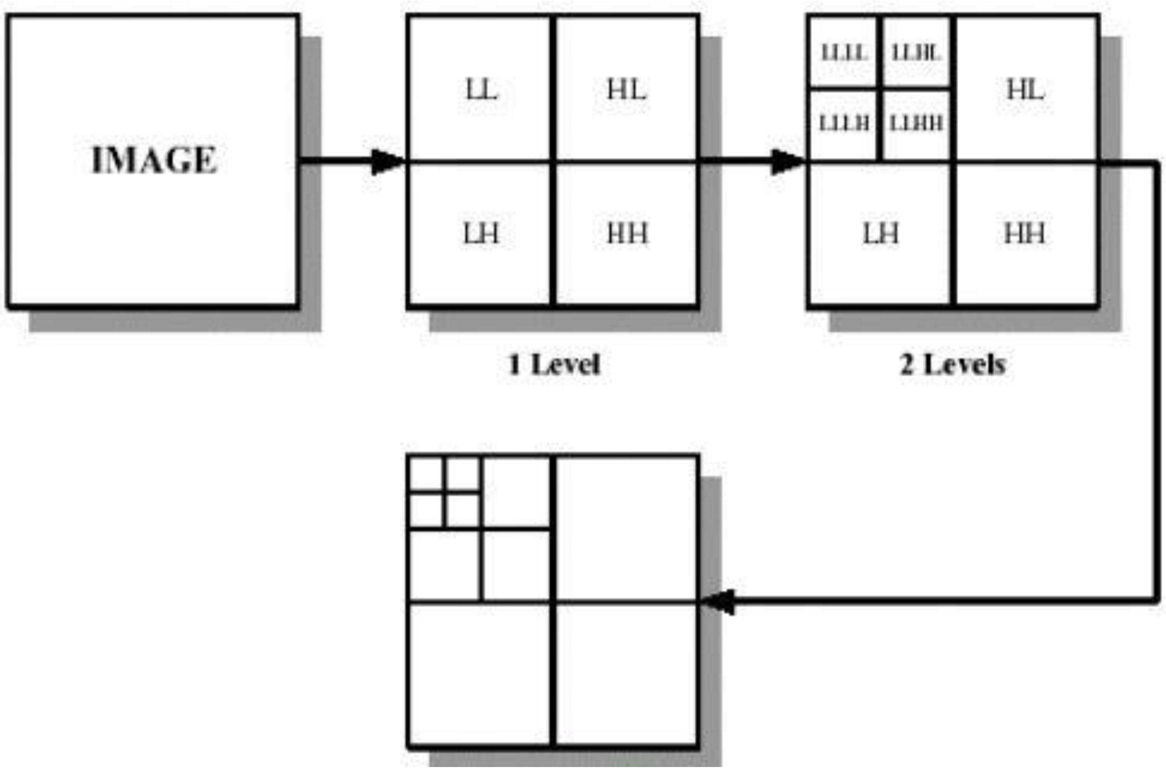

Stage 4 lo
5
0
-5
0
5
10
15
20
25
30
35
stage 5 - hi
4
2
0
-2
0
5
10
15
20
25
30
stage 6 - lo
2
0
-2
0
5
10
15
20
25
30


Stage 4 lo
2
0
-2
0
5
10
15
stage 5 - hi
0.5
0
-0.5
-1
0
2
4
6
8
10
12
14
stage 6 - lo
1
0
-1
-2
0
2
4
6
8
10
12
14

Had enough?

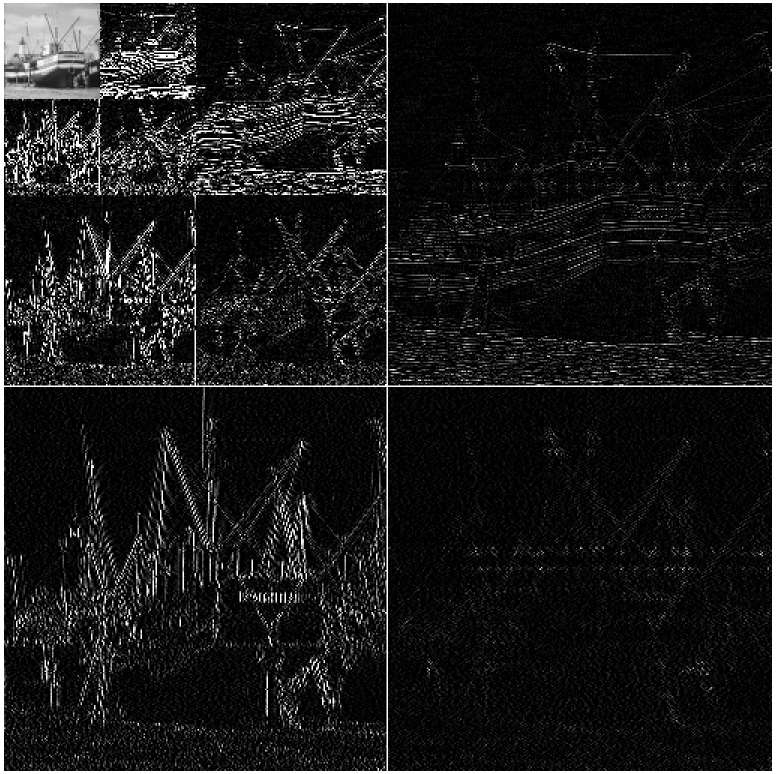
Discrete Wavelet Transform

- **2-D DWT for Image**



2-D WT Example

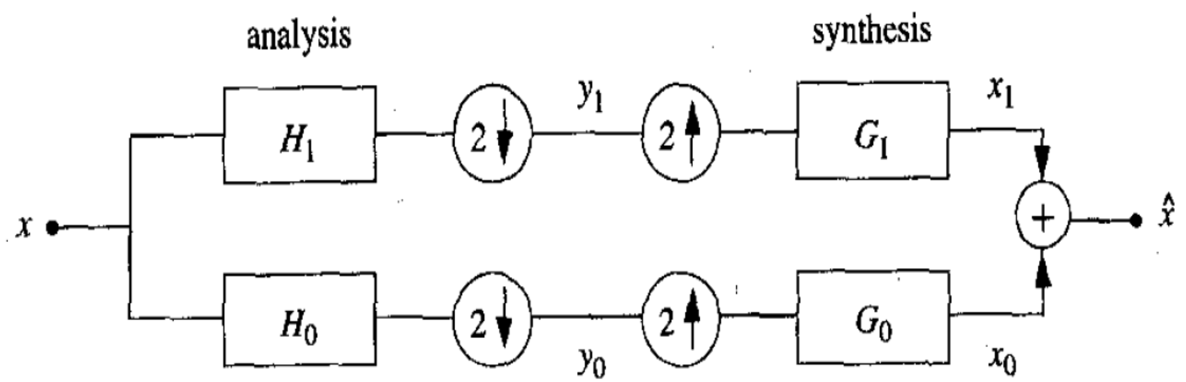
Boats image



WT in 3 levels

Two-Channel Filter Banks

- Filter bank is the building block of discrete-time wavelet transform
- For 1-D signals, two-channel filter bank is depicted below



Two-Channel Filter Banks

- For perfect reconstruction filter banks we have

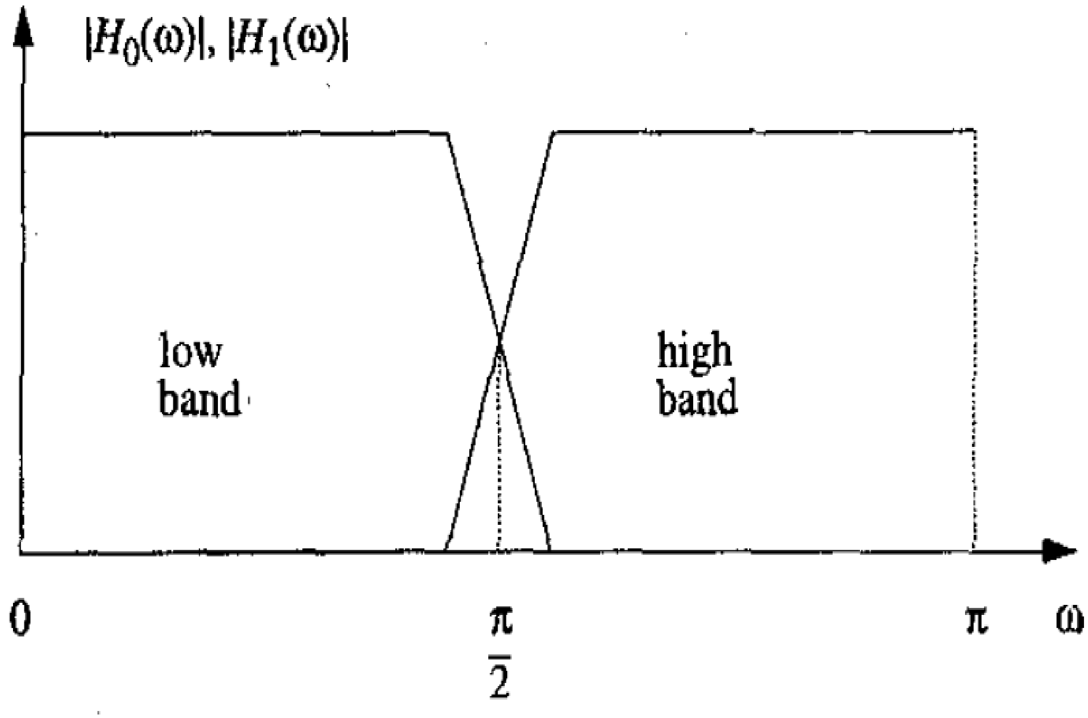
$$\hat{x} = x$$

- In order to achieve perfect reconstruction the filters should satisfy

$$\begin{cases} g_0[n] = -h_0[-n] \\ g_1[n] = h_1[-n] \end{cases}$$

- Thus if one filter is lowpass, the other one will be highpass

Two-Channel Filter Banks



Discrete Wavelet Transform

- We can construct discrete WT via iterated (octave-band) filter banks
- The analysis section is illustrated below